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ON A SIMPLE WAY OF SIMULATING WAVE PROPAGATION IN COMPLEX GEOMETRIES

Koji Uenishi

Abstract: This contribution introduces a simple numerical approach for wave simulations based on the method of characteristics. The numerical formulation is for one- and two-dimensional linear elastic media with/without interfaces between dissimilar materials. We shall consider the information about physical quantities themselves as well as their first spatial derivatives, and shift that information with sound velocities for the next time step. The method may be applied especially in the field of seismology, solid mechanics as well as civil and mechanical engineering.

Key words: Wave propagation, method of characteristics, boundary conditions, complex geometry

1. INTRODUCTION

Recent advances in computational technologies have made it possible to simulate rather complex physical phenomena even on a PC basis, and typical numerical methods used in solid mechanics are finite difference method (FDM), finite element method (FEM) and boundary element method (BEM). For wave and fracture dynamics, FDM is computationally efficient but the geometry employed must be generally rather simple. If we have complex geometries, FEM may be more appropriate, but caution should be exercised regarding its mesh dependency. If we have infinitely extending media, then BEM may be more suitable, but we do need boundary integral equation (BIE) formulations usually involving some mathematical treatment of singularities. In this study, we consider problems of wave dynamics based on the method of characteristics. Our point here is that values of physical properties as well as their first spatial derivatives are taken into account in the numerical formulation in order to maintain more precisely the time-dependent distributions of stresses and particle velocities in linear elastic media. At first sight, the method resembles FDM, but in reality, the new formulation is not influenced by positions of grid points and therefore we can simulate more complicated wave problems with more complex geometrical conditions efficiently. We shall consider one- and two-dimensional cases with (or without) interfaces between dissimilar materials and free (rigid) surfaces.
2. ONE-DIMENSIONAL CASE

(1) Wave propagation in an infinite, monolithic linear elastic medium

Consider the equation of motion and the stress (σ)-displacement (u) relation for a one-dimensional infinite, monolithic linear elastic medium:\(^{4,5}\):

\[
\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x},
\]

\[
\sigma = E \frac{\partial u}{\partial x}.
\]

(1)

Here, \(\rho\) is mass density, \(E\) is Young's modulus, \(t\) is time and the wave is propagating in the \(x\)-direction. By introducing the generalized particle velocity \(\dot{U} = \rho \frac{\partial u}{\partial t}\) with the wave speed in the medium \(c = \sqrt{E/\rho}\), and assuming the material properties (\(\rho, E\)) do not vary in time and space, we can rewrite the above equations as

\[
\frac{\partial}{\partial t} \begin{pmatrix} \dot{U} \\ \sigma \end{pmatrix} + \begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} \dot{U} \\ \sigma \end{pmatrix} = \mathbf{0},
\]

or,

\[
\frac{\partial}{\partial t} \begin{pmatrix} \dot{U} + \sigma \\ \dot{U} - \sigma \end{pmatrix} + \begin{pmatrix} -c & 0 \\ 0 & +c \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} \dot{U} + \sigma \\ \dot{U} - \sigma \end{pmatrix} = \mathbf{0}.
\]

(3)

Similarly, for the \(x\)-derivative,

\[
\frac{\partial}{\partial t} \left( \frac{\partial \dot{U}}{\partial x} + \frac{\partial \sigma}{\partial x} \right) + \begin{pmatrix} -c & 0 \\ 0 & +c \end{pmatrix} \frac{\partial}{\partial x} \left( \frac{\partial \dot{U}}{\partial x} + \frac{\partial \sigma}{\partial x} \right) = \mathbf{0},
\]

(4)

holds. Equations (3) and (4) indicate that the information about \(\dot{U} \pm \sigma\) and \(\partial \dot{U}/\partial x \pm \partial \sigma/\partial x\) propagates with the velocity \(\mp c\), respectively: if we would like to know \(\dot{U} \pm \sigma\) \((\partial \dot{U}/\partial x \pm \partial \sigma/\partial x)\) at a point \(X\) for the time \(t = T\), as shown in Figure 1, we just respectively shift the values of \(\dot{U} \pm \sigma\) \((\partial \dot{U}/\partial x \pm \partial \sigma/\partial x)\) at \((X \mp c\Delta t, T - \Delta t)\). Namely,

\[
\dot{U}(X, T) = \frac{\dot{U}(X - c\Delta t, T - \Delta t) + \dot{U}(X + c\Delta t, T - \Delta t)}{2} - \sigma(X - c\Delta t, T - \Delta t) + \sigma(X + c\Delta t, T - \Delta t))/2,
\]

\[
\sigma(X, T) = \frac{\sigma(X - c\Delta t, T - \Delta t) + \sigma(X + c\Delta t, T - \Delta t)}{2},
\]

\[
\frac{\partial \dot{U}(X, T)}{\partial x} = \frac{\partial \dot{U}(X - c\Delta t, T - \Delta t)}{\partial x} + \partial \dot{U}(X + c\Delta t, T - \Delta t)/(\partial x)
\]

\[
- \partial \sigma(X - c\Delta t, T - \Delta t))/(\partial x) \pm \partial \sigma(X + c\Delta t, T - \Delta t)/(\partial x)\\/2,
\]

\[
\frac{\partial \sigma(X, T)}{\partial x} = \frac{\partial \sigma(X - c\Delta t, T - \Delta t)}{\partial x} \pm \partial \sigma(X + c\Delta t, T - \Delta t)/(\partial x)\!
\]

(5)

Thus we can obtain the values of \(\dot{U}, \sigma, \partial \dot{U}/\partial x\) and \(\partial \sigma/\partial x\) at \(t = T\) from their associated values at \(t = T - \Delta t\) and numerically construct the spatial particle velocity/stress distributions effectively.

(2) Boundary conditions

We have considered above a monolithic medium. If there is an interface at \(x = X\) between two dissimilar materials, 1 and 2 [Figure 2(a)], then the boundary conditions at \(x = X\) are given by
Figure 1 One-dimensional wave propagation. The information about $U \pm \sigma$ propagates along characteristic lines with the velocity $\pm c$.

Figure 2 One-dimensional wave interaction with (a) an interface between two dissimilar materials; and (b) a free/rigid surface.

$$\dot{u}_1 = \dot{u}_2 = \dot{u}, \quad \sigma_1 = \sigma_2 = \sigma. \quad (6)$$

Here, $(\ast) = \partial(\cdot)/\partial t$, and the subscripts 1 and 2 correspond to the properties of materials 1 and 2, respectively. By defining the generalized particle velocities $\dot{U}_1 = \rho_1 c_1 \dot{u}_1$ and $\dot{U}_2 = \rho_2 c_2 \dot{u}_2$, as well as $\dot{U}_1' = \dot{U}_1(X - c_1 \Delta t, T - \Delta t)$, $\sigma_1' = \sigma_1(X - c_1 \Delta t, T - \Delta t)$, $\dot{U}_2 = \dot{U}_2(X + c_2 \Delta t, T - \Delta t)$ and $\sigma_2 = \sigma_2(X + c_2 \Delta t, T - \Delta t)$, we have from equations (3) and (6)

$$\rho_1 c_1 \dot{u}(T) - \sigma(T) = \dot{U}_1' - \sigma_1' = A, \quad \rho_2 c_2 \dot{u}(T) + \sigma(T) = \dot{U}_2 + \sigma_2 = B, \quad (7)$$

or

$$\dot{u}(T) = (A + B)/(\rho_1 c_1 + \rho_2 c_2), \quad \sigma(T) = (\rho_1 c_1 B - \rho_2 c_2 A)/(\rho_1 c_1 + \rho_2 c_2), \quad (8)$$

and finally we obtain

$$\dot{U}_1(X, T) = \rho_1 c_1 \dot{u}(T) = \rho_1 c_1 (A + B)/(\rho_1 c_1 + \rho_2 c_2),$$

$$\dot{U}_2(X, T) = \rho_2 c_2 \dot{u}(T) = \rho_2 c_2 (A + B)/(\rho_1 c_1 + \rho_2 c_2),$$

$$\sigma_1(X, T) = \sigma_1(X, T) = \sigma(T) = -(\rho_2 c_2 A + \rho_1 c_1 B)/(\rho_1 c_1 + \rho_2 c_2). \quad (9)$$
If material 2 does not exist (and we omit the subscript 1 for the material 1), then
\[ U(X, T) - \sigma(X, T) = U(X - c\Delta t, T - \Delta t) - \sigma(X - c\Delta t, T - \Delta t) = U^* - \sigma^* = A^* \] [Figure 2(b)]. For a free boundary (\( \sigma = 0 \) at \( x = X \)), \( U^*(X, T) = \dot{A}^* \) holds, and for a rigid boundary (\( \dot{U} = 0 \) at \( x = X \)), we have \( \sigma(X, T) = -A^* \).

For the x-derivatives [Figure 3(a)], similarly, the boundary conditions at \( x = X \) are
\[ \dot{U}_1 = \dot{u}_2, \quad \text{and} \quad \sigma_1 = \sigma_2, \] (10)
where \( (\ddot{}') \equiv \partial^2(\cdot)/\partial^2. \) They are equivalent to
\[ (1/\rho_1) \partial \sigma_1 / \partial x = (1/\rho_2) \partial \sigma_2 / \partial x, \quad \text{and} \quad c_1 \partial U_1 / \partial x = c_2 \partial U_2 / \partial x, \] (11)
and from equation (5), we have
\[ \partial U_1 / \partial x - \partial \sigma_1 / \partial x = \partial (U_1 - \sigma_1) / \partial x = C, \quad \text{and} \quad \partial U_2 / \partial x + \partial \sigma_2 / \partial x = \partial (U_2 + \sigma_2) / \partial x = D. \] (12)

Therefore, by combining equations (11) and (12), we obtain
\[
\partial U_1 / \partial x = [p_1 c_1 c + p_2 c_2 (c_1 / c_2) D] / (p_1 c_1 + p_2 c_2), \\
\partial U_2 / \partial x = (c_1 / c_2) \partial U_1 / \partial x = [p_2 c_2 c_1 / c_2 D] / (p_1 c_1 + p_2 c_2), \\
\partial \sigma_1 / \partial x = [-p_1 c_1 C + p_2 c_2 (c_1 / c_2) D] / (p_1 c_1 + p_2 c_2), \\
\partial \sigma_2 / \partial x = (p_2 / p_1) \partial \sigma_1 / \partial x = [-p_2 c_2 c_1 / c_2 C + p_2 c_2 D] / (p_1 c_1 + p_2 c_2). \] (13)

If there is no material 2 (and again by omitting the subscript 1) [Figure 3(b)], the condition \( \partial U / \partial x = \partial \sigma / \partial x = \partial (U^* - \sigma^*) / \partial x = C^* \) gives \( \partial \sigma / \partial x = -C^* \) for a free boundary (\( \sigma = 0 \) or \( p c^2 \partial \dot{\sigma} / \partial x = 0 \), i.e., \( \partial U / \partial x = 0 \)) and \( \partial U / \partial x = C^* \) for a rigid boundary [\( \dot{u} = 0 \) or \( (1/p) \partial \sigma / \partial x = 0 \).]
3. TWO-DIMENSIONAL CASE

(1) Wave propagation in an infinite, monolithic linear elastic medium

Now consider the equations of motion and the stress-displacement relations for a two-dimensional infinite, monolithic linear elastic medium\textsuperscript{4,5}:

\[
\begin{align*}
\rho \frac{\partial^2 u}{\partial t^2} &= \sigma_{x}\frac{\partial}{\partial x} + \sigma_{xy}\frac{\partial}{\partial y}, \\
\rho \frac{\partial^2 v}{\partial t^2} &= \sigma_{y}\frac{\partial}{\partial x} + \sigma_{xy}\frac{\partial}{\partial y}, \\
\sigma_{x} &= (\lambda + 2\mu) \frac{\partial u}{\partial x} + \lambda \frac{\partial v}{\partial y}, \\
\sigma_{y} &= \lambda \frac{\partial u}{\partial x} + (\lambda + 2\mu) \frac{\partial v}{\partial y}, \\
\tau_{xy} &= \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right),
\end{align*}
\]

Here, \( \lambda \) and \( \mu \) are time-independent Lamé\'s constants, \( u \) (\( v \)) is the displacement in the \( x \)- (\( y \)-) direction, respectively, \( \sigma_{x} \) and \( \sigma_{y} \) are the normal stresses, and \( \tau_{xy} \) is the shear stress in the \( xy \)-plane. Defining the generalized particle velocities \( \vec{U} = \rho c_{p} \vec{u} \) and \( \vec{V} = \rho c_{s} \vec{v} \), and using the relation \( \lambda + 2\mu = \rho c_{p}^{2} \) and \( \mu = \frac{\rho c_{s}^{2}}{c_{p}} \) (\( c_{p} \): P-wave speed, \( c_{s} \): S-wave speed), we can derive from the above equations the relation

\[
\begin{align*}
\frac{\partial}{\partial t} \begin{pmatrix} \vec{U} \\ \vec{V} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & c_{p} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{p} \\
c_{p} & 0 & 0 & 0 & 0 \\
c_{p} - 2c_{s}^{2}/c_{p} & 0 & 0 & 0 & 0 \\
c_{s}^{2}/c_{p} & 0 & 0 & 0 & 0 \\
\end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} \vec{U} \\ \vec{V} \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & 0 & 0 & c_{p} \\
0 & 0 & 0 & 0 & c_{p} \\
0 & c_{p} - 2c_{s}^{2}/c_{p} & 0 & 0 & 0 \\
0 & c_{s}^{2}/c_{p} & 0 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix} \vec{U} \\ \vec{V} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & c_{p} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{p} \\
c_{p} & 0 & 0 & 0 & 0 \\
c_{p} - 2c_{s}^{2}/c_{p} & 0 & 0 & 0 & 0 \\
c_{s}^{2}/c_{p} & 0 & 0 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix} \vec{U} \\ \vec{V} \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & 0 & 0 & c_{p} \\
0 & 0 & 0 & 0 & c_{p} \\
0 & c_{p} - 2c_{s}^{2}/c_{p} & 0 & 0 & 0 \\
0 & c_{s}^{2}/c_{p} & 0 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix} \vec{U} \\ \vec{V} \end{pmatrix}.
\end{align*}
\]

Equation (15) may be regarded as the superposition of

\[
\begin{align*}
\frac{\partial}{\partial t} \begin{pmatrix} \vec{U} \\ \vec{V} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & c_{p} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{p} \\
c_{p} & 0 & 0 & 0 & 0 \\
c_{p} - 2c_{s}^{2}/c_{p} & 0 & 0 & 0 & 0 \\
c_{s}^{2}/c_{p} & 0 & 0 & 0 & 0 \\
\end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} \vec{U} \\ \vec{V} \end{pmatrix}, \\
\end{align*}
\]

in the \( x \)-direction and

\[
\begin{align*}
\frac{\partial}{\partial t} \begin{pmatrix} \vec{U} \\ \vec{V} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 & 0 & c_{p} \\
0 & 0 & c_{p} & 0 & 0 \\
0 & c_{p} & 0 & 0 & 0 \\
c_{p} - 2c_{s}^{2}/c_{p} & 0 & 0 & 0 & 0 \\
c_{s}^{2}/c_{p} & 0 & 0 & 0 & 0 \\
\end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} \vec{U} \\ \vec{V} \end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
\sigma_{s} &= (c_{p} - 2c_{s}^{2}/c_{p}) \frac{\partial \vec{U}}{\partial x} = \left( 1 - 2c_{s}^{2}/c_{p}^{2} \right) \sigma_{s} = \frac{\nu}{1-\nu} \sigma_{s}, \\
\sigma_{s} &= (c_{p} - 2c_{s}^{2}/c_{p}) \frac{\partial \vec{V}}{\partial y} = \frac{\nu}{1-\nu} \sigma_{s},
\end{align*}
\]
in the y-direction. From equation (16), we obtain

\[
\frac{\partial}{\partial t} \left( \tilde{U} + \sigma_x \right) + \left( -c_p \right) \frac{\partial}{\partial x} \left( \tilde{U} - \sigma_x \right) = 0,
\]

\[
\frac{\partial}{\partial t} \left( \tilde{V} + \frac{c_p}{c_s} \tau_{xy} \right) + \left( -c_s \right) \frac{\partial}{\partial y} \left( \tilde{V} - \frac{c_p}{c_s} \tau_{xy} \right) = 0,
\]

and the last equation (18) indicates that in the positive (negative) x-direction the information about \( \tilde{U} - \sigma_x \) \((\tilde{U} + \sigma_x)\) propagates at the P-wave speed \( c_p \) while \( \tilde{V} - \frac{c_p}{c_s} \tau_{xy} \) \( [\tilde{V} + \frac{c_p}{c_s} \tau_{xy}] \) moves at the S-wave speed \( c_s \) respectively. Note that \( \frac{c_p}{c_s} = \sqrt{2(1-\nu)/(1-2\nu)} \) depends only on Poisson’s ratio in the medium. Similarly, from equation (17), we have for the y-direction

\[
\frac{\partial}{\partial t} \left( \tilde{V} + \sigma_y \right) + \left( -c_p \right) \frac{\partial}{\partial x} \left( \tilde{V} - \sigma_y \right) = 0,
\]

\[
\frac{\partial}{\partial t} \left( \tilde{U} + \frac{c_p}{c_s} \tau_{xy} \right) + \left( -c_s \right) \frac{\partial}{\partial y} \left( \tilde{U} - \frac{c_p}{c_s} \tau_{xy} \right) = 0.
\]

Expressions similar to equations (18) and (19) can be obtained for the x- and y- derivatives, respectively.

(2) Boundary conditions

For the two-dimensional case, we first consider a vertical interface between two different materials, 1 and 2, located at \( x = X \) [Figure 4(a)]. Horizontal interfaces can be treated similarly, and in the next section we shall study the conditions for inclined boundaries.

At a point on the boundary interface, \((X, Y)\), the following conditions apply:

\[
\tilde{u} = \tilde{u}_2 = \tilde{u}_2, \quad \tilde{v} = \tilde{v}_2 = \tilde{v}_2, \quad \sigma_n = \sigma_n = \sigma_n, \quad \text{and} \quad \tau_{xy} = \tau_{y_2} = \tau_{y_2}.
\]

By introducing \( \tilde{U}_1 = \rho_1 c_{p1} \tilde{u}_1, \quad \tilde{V}_1 = \rho_1 c_{p1} \tilde{v}_1, \quad \tilde{U}_2 = \rho_2 c_{p2} \tilde{u}_2, \quad \tilde{V}_2 = \rho_2 c_{p2} \tilde{v}_2, \quad \tilde{U}'_1 = \tilde{U}_1(X - c_{p1} \Delta t, Y, T - \Delta t), \quad \tilde{V}'_1 = \tilde{V}_1(X - c_{p1} \Delta t, Y, T - \Delta t), \quad \sigma_{n1} = \sigma_{n1}(X - c_{p1} \Delta t, Y, T - \Delta t), \quad \tau_{y1} = \tau_{y_1}(X - c_{p1} \Delta t, Y, T - \Delta t), \quad \tilde{U}'_2 = \tilde{U}_2(X + c_{p2} \Delta t, Y, T - \Delta t), \quad \tilde{V}'_2 = \tilde{V}_2(X + c_{p2} \Delta t, Y, T - \Delta t), \quad \sigma_{n2} = \sigma_{n2}(X + c_{p2} \Delta t, Y, T - \Delta t), \quad \text{and finally} \quad \tau_{y2} = \tau_{y_2}(X + c_{p2} \Delta t, Y, T - \Delta t), \quad \text{we have from equation (18)} \),

\[
\rho_1 c_{p1} \tilde{u}(T) - \sigma_{n1}(T) = \tilde{U}'_1 - \sigma_{n1} = A,
\]

\[
\rho_1 c_{p2} \tilde{u}(T) + \sigma_{n2}(T) = \tilde{U}'_2 + \sigma_{n2} = B,
\]

\[
\rho_1 c_{p1} \tilde{v}(T) - \left( c_{p1}/c_{s1} \right) \tau_{y1}(T) = \tilde{V}'_1 - \left( c_{p1}/c_{s1} \right) \tau_{y1} = C,
\]

\[
\rho_2 c_{p2} \tilde{v}(T) + \left( c_{p2}/c_{s2} \right) \tau_{y2}(T) = \tilde{V}'_2 + \left( c_{p2}/c_{s2} \right) \tau_{y2} = D.
\]

or
Thus we obtain

\[
\begin{align*}
\bar{U}_1(X,Y,T) &= \frac{1}{\rho_1} \left[ \frac{(c_{31}/\rho_1) + (c_{32}/\rho_2) D}{\rho_1 c_{31} + \rho_2 c_{32}} \right], \\
\bar{V}_1(X,Y,T) &= \frac{1}{\rho_1} \left[ \frac{(c_{31}/\rho_1) + (c_{32}/\rho_2) D}{\rho_1 c_{31} + \rho_2 c_{32}} \right], \\
\bar{U}_2(X,Y,T) &= \frac{1}{\rho_2} \left[ \frac{(c_{31}/\rho_1) + (c_{32}/\rho_2) D}{\rho_1 c_{31} + \rho_2 c_{32}} \right], \\
\bar{V}_2(X,Y,T) &= \frac{1}{\rho_2} \left[ \frac{(c_{31}/\rho_1) + (c_{32}/\rho_2) D}{\rho_1 c_{31} + \rho_2 c_{32}} \right], \\
\sigma_{\alpha \beta}(X,Y,T) &= \sigma_{\alpha \beta}(X,Y,T) = \sigma_{\alpha \beta}(T) = \left[ \frac{1}{\rho_1} + \frac{1}{\rho_2} \right], \\
\tau_{\alpha \beta}(X,Y,T) &= \tau_{\alpha \beta}(X,Y,T) = \tau_{\alpha \beta}(T) = \left[ \frac{1}{\rho_1} + \frac{1}{\rho_2} \right].
\end{align*}
\]

If this approximation does not provide sufficiently appropriate results, the information \( \bar{U}_1^*, \) etc., may be set, for example, like \( \bar{U}_1^* = \left[ \frac{1}{\rho_1} \left( X - c_{\alpha \beta} \Delta t, Y - c_{\alpha \beta} \Delta t, T - \Delta t \right) + \frac{1}{\rho_1} \left( X - c_{\alpha \beta} \Delta t, Y - c_{\alpha \beta} \Delta t, T - \Delta t \right) \right] \) \( \tau_{\alpha \beta}(T) = \left[ \frac{1}{\rho_1} + \frac{1}{\rho_2} \right]. \) [Figure 4(b)]. For a free boundary (\( \bar{U}_1^* = 0 \)), \( \bar{U}_1(X,Y,T) = \bar{A} \) and \( \bar{V}_1(X,Y,T) = \bar{B} \), while for a rigid boundary (\( \bar{U}_1^* = \bar{V}_1 = 0 \)), we obtain \( \sigma_{\alpha \beta}(X,Y,T) = \sigma_{\alpha \beta}(T) = \left[ \frac{1}{\rho_1} + \frac{1}{\rho_2} \right], \) \( \tau_{\alpha \beta}(X,Y,T) = \tau_{\alpha \beta}(T) = \left[ \frac{1}{\rho_1} + \frac{1}{\rho_2} \right]. \)

Similarly, for the \( x \)-derivatives [Figure 5(a)], the boundary conditions at \( x = X, y = Y \) are

\[
\bar{u}_1 = \bar{v}_1, \quad \bar{v}_1 = \bar{v}_2, \quad \bar{u}_1 = \bar{u}_2, \quad \text{and} \quad \bar{v}_1 = \bar{v}_2,
\]

which are equivalent to

\[
\begin{align*}
\frac{1}{\rho_1} \left( \frac{\partial \sigma_{\alpha \beta}}{\partial x} + \frac{\partial \bar{U}_1}{\partial x} + \frac{\partial \bar{U}_2}{\partial x} + \partial \tau_{\alpha \beta} \right) &= \frac{1}{\rho_1} \left( \frac{\partial \sigma_{\alpha \beta}}{\partial x} + \frac{\partial \bar{V}_1}{\partial x} + \frac{\partial \bar{V}_2}{\partial x} + \partial \tau_{\alpha \beta} \right), \\
\frac{1}{\rho_1} \left( \frac{\partial \tau_{\alpha \beta}}{\partial x} + \frac{\partial \bar{U}_1}{\partial x} + \frac{\partial \bar{U}_2}{\partial x} + \partial \tau_{\alpha \beta} \right) &= \frac{1}{\rho_1} \left( \frac{\partial \tau_{\alpha \beta}}{\partial x} + \frac{\partial \bar{V}_1}{\partial x} + \frac{\partial \bar{V}_2}{\partial x} + \partial \tau_{\alpha \beta} \right), \\
C_{\rho_1} \left( \frac{\partial \bar{U}_1}{\partial x} + \frac{\partial \bar{V}_1}{\partial x} + \frac{\partial \bar{U}_2}{\partial x} + \partial \tau_{\alpha \beta} \right) &= C_{\rho_1} \left( \frac{\partial \bar{U}_1}{\partial x} + \frac{\partial \bar{V}_1}{\partial x} + \frac{\partial \bar{U}_2}{\partial x} + \partial \tau_{\alpha \beta} \right), \\
C_{\rho_1} \left( \frac{\partial \bar{V}_1}{\partial x} + \frac{\partial \bar{U}_1}{\partial x} + \frac{\partial \bar{U}_2}{\partial x} + \partial \tau_{\alpha \beta} \right) &= C_{\rho_1} \left( \frac{\partial \bar{V}_1}{\partial x} + \frac{\partial \bar{U}_1}{\partial x} + \frac{\partial \bar{U}_2}{\partial x} + \partial \tau_{\alpha \beta} \right).
\end{align*}
\]
Figure 5  Boundary conditions for the $x$-derivatives related to two-dimensional waves: (a) an interface between two different materials; and (b) a free/rigid surface.

Referring to equation (18), we have approximately

$$
\frac{\partial U_1}{\partial x} - \frac{\partial \sigma_{1z}}{\partial x} = \frac{\partial (U_1 - \sigma_{1z})}{\partial x} = E, \\
\frac{\partial U_2}{\partial x} + \frac{\partial \sigma_{2z}}{\partial x} = \frac{\partial (U_2 + \sigma_{2z})}{\partial x} = F, \\
\frac{\partial V_1}{\partial x} - (c_{p1}/v_{c1}) \frac{\partial \tau_{y1}}{\partial x} = \frac{\partial (V_1 - (c_{p1}/v_{c1}) \tau_{y1})}{\partial x} = G, \\
\frac{\partial V_2}{\partial x} + (c_{p2}/v_{c2}) \frac{\partial \tau_{y2}}{\partial x} = \frac{\partial (V_2 + (c_{p2}/v_{c2}) \tau_{y2})}{\partial x} = H, \\
$$

(26)

Since the $y$-derivatives in equation (25) are known approximately near the boundary, we write them as

$$
\frac{\partial U_1}{\partial y} = U_1^*, \quad \frac{\partial V_1}{\partial y} = V_1^*, \quad \frac{\partial \sigma_{1z}}{\partial y} = \sigma_{1z}^*, \quad \frac{\partial \sigma_{2z}}{\partial y} = \sigma_{2z}^*, \quad \frac{\partial \tau_{y1}}{\partial y} = \tau_{y1}^*, \quad \frac{\partial \tau_{y2}}{\partial y} = \tau_{y2}^*. 
$$

By combining equations (25) and (26), and defining $F' = F - (p_2/p_1)(c_{p1}/c_{p2})(1 - 2(c_{p1}/v_{c1})V_{1y}^*) + \tau_{y2}^* + 1 - 2(c_{p2}/c_{p2})V_{2y}^*$ and $H' = H - (p_2/p_1)(c_{p1}/c_{p2})\sigma_{1z}^* - (c_{p2}/v_{c2})(c_{p2}/c_{p2})U_{1y}^* + (c_{p2}/c_{p2})\sigma_{2z}^* + U_{2y}^*$, we obtain

$$
\frac{\partial U_1}{\partial x} = \frac{\partial V_1}{\partial x} = \frac{\partial \sigma_{1z}}{\partial x} = \frac{\partial \sigma_{2z}}{\partial x} = \frac{\partial \tau_{y1}}{\partial x} = \frac{\partial \tau_{y2}}{\partial x} = 0, \\
$$

(27)

If we have no material 2 (and again omit the subscript 1) [Figure 5(b)], the conditions

$$
\frac{\partial \bar{U}}{\partial x} - \frac{\partial \sigma_t}{\partial x} = \frac{\partial (U^* - \sigma_t)}{\partial x} = E^* \quad \text{and} \quad \frac{\partial \bar{V}}{\partial x} - (c_{p1}/v_{c1}) \frac{\partial \tau_y}{\partial x} = \frac{\partial (V^* - (c_{p1}/v_{c1}) \tau_y)}{\partial x} = G^* 
$$

render $\frac{\partial \sigma_t}{\partial x} = -E^* - (1 - 2c_{p1}^2/c_{p1})V_{1y}^*$ and $\frac{\partial \tau_y}{\partial x} = -(c_{p1}/v_{c1}) (G^* + U_{1y}^*)$ for a free boundary ($\sigma_t = \tau_y = 0$), and

$$
\frac{\partial \bar{U}}{\partial x} = E^* - \tau_{y2}^* \quad \text{and} \quad \frac{\partial \bar{V}}{\partial x} = G^* - (c_{p1}/v_{c1}) \sigma_{2z}^* 
$$

for a rigid boundary ($\bar{u} = \bar{v} = 0$), with known ($\partial \bar{U}/\partial y = \bar{U}_{1y}^*$, $\partial \bar{V}/\partial y = \bar{V}_{1y}^*$) and ($\partial \sigma_t/\partial y = \sigma_{1z}^*$, $\partial \tau_y/\partial y = \tau_{y1}^*$), respectively.
In this last section, we consider more complex geometry and mathematical conditions for an inclined boundary (Figure 6). Assume the inclination of the interface is \( \beta \). If we consider the case \( \pi/4 < \beta < \pi/2 \), for example, the boundary conditions that the normal and tangential displacements and tractions are continuous imply

\[
\begin{pmatrix}
\rho \frac{c_{x^2}}{c_{x^2}} & 0 & \rho \frac{c_{x^1}}{c_{x^1}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho \frac{c_{x^2}}{c_{x^2}} & 0 & \rho \frac{c_{x^1}}{c_{x^1}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sin^2 \beta + \frac{v_1}{1-v_1} \cos^2 \beta & -2\sin \beta \cos \beta & -\sin^2 \beta - \frac{v_1}{1-v_1} \cos^2 \beta & 2\sin \beta \cos \beta & 2\sin \beta \cos \beta \\
0 & 0 & 0 & 0 & -1-2v_1 \sin \beta \cos \beta & \cos^2 \beta - \sin^2 \beta & 1-2v_2 \sin \beta \cos \beta & -\cos^2 \beta + \sin^2 \beta & 1-2v_2 \sin \beta \cos \beta \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -c_{x^1}/c_{x^1} & 0 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
U_1(X,Y,T) \\
V_1(X,Y,T) \\
\dot{U}_1(X,Y,T) \\
\dot{V}_1(X,Y,T) \\
\Delta \sigma_{x^1} \\
\tau_{x^1}(X,Y,T) \\
\Delta \sigma_{x^2} \\
\tau_{x^2}(X,Y,T)
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
\sigma_{x^1}(X,Y,T) = \sigma_{x^1}^* + \Delta \sigma_{x^1}, \\
\sigma_{x^2}(X,Y,T) = \sigma_{x^2}^* + \Delta \sigma_{x^2}, \\
\sigma_{y^1}(X,Y,T) = \sigma_{y^1}^* + v_1/(1-v_1) \Delta \sigma_{x^1}, \\
\sigma_{y^2}(X,Y,T) = \sigma_{y^2}^* + v_2/(1-v_2) \Delta \sigma_{x^2}
\end{pmatrix}
\]

Here, \( (\cdot)^* \) denotes the values as defined before. Similarly, we can obtain the expressions for other cases of \( \beta \) as well as for the spatial derivatives.

Figure 6  Two-dimensional wave interaction with an inclined interface between two dissimilar materials.
4. CONCLUSIONS

We have derived some mathematical expressions for wave propagation in linear elastic materials using the method of characteristics. For both one- and two-dimensional cases, we can show that the necessary information about the physical quantities (generalized particle velocities and stresses) and their first spatial derivatives propagate at sound velocities, and thus we can construct the distributions of those quantities efficiently for the next time step. The information at the previous time step can be taken virtually from every place in the media and therefore, restrictions regarding the positions of grid points and boundary geometry are lessened compared with, e.g., finite difference method.

REFERENCES


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