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Transverse Instability of Solitary Waves in the Generalized Kadomtsev-Petviashvili Equation

Takeshi Kataoka,* Michihisa Tsutahara, and Yoshihiro Negoro

Graduate School of Science and Technology, Kobe University, Rokkodai, Nada, Kobe 657–8501, Japan

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The linear stability of planar solitary waves with respect to long-wavelength transverse perturbations is studied in the framework of the generalized Kadomtsev-Petviashvili equation. It is newly discovered that for some nonlinearities in this family, the solitary waves could be transversely unstable even in a medium with negative dispersion. In the case of positive dispersion, they are found to be always unstable.

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When we study the nonlinear behavior of long small-amplitude waves propagating in weakly dispersive media, the Korteweg–de Vries (KdV) equation [1], which is derived by the weakly nonlinear analysis, is often used as a model equation. The KdV equation contains the leading-order terms representing nonlinearity and wave dispersion, and it has been found that this equation can predict reasonably well the one-dimensional behavior of long nonlinear dispersive waves, such as planar solitary waves, of small, but not infinitesimal, amplitude. In contrast, if wave amplitude is not supposed to be small, the following generalized Korteweg–de Vries (GKdV) equation often arises as an approximate model [2–5]:

\[ u_t + [f(u)]_x + u_{xxx} = 0 , \]

where \( f(u) \) is a nonlinear function of \( u \), represents the effects of nonlinearity. The usual KdV equation occurs for \( f(u) = cu^2 \) with \( c \) constant. The subscripts \( x \) and \( t \) denote partial differentiations. This notation will be used hereafter.

Various generalizations of the above KdV-type equation to multidimensions are possible [6,7]. Here we consider waves propagating in two space dimensions with weak dependence on the transverse (\( y \)) direction. Then their behavior is described by the generalized Kadomtsev-Petviashvili (GKP) equation,

\[ (u_t + [f(u)]_x + u_{xxx})_x - \frac{\sigma}{2} u_{yy} = 0 , \]

which corresponds to the cases of positive and negative dispersion when \( \sigma = 1 \) and \( -1 \), respectively. For waves in fluids, the positive dispersion case arises for describing shallow water waves where surface tension dominates, while the negative dispersion case arises for describing any long gravity waves, such as internal waves and water waves with little surface tension effects. Therefore the latter is considered to be the more universal case we often encounter in describing long nonlinear dispersive waves in fluids.

The purpose of this Letter is to conduct the linear stability analysis of planar solitary waves with respect to long-wavelength transverse perturbations on the basis of the GKP equation (2). Although we examine the stability for both cases of dispersion, our new finding exists in the case of negative dispersion. Before mentioning our results specifically, we first outline the preceding studies concerning the stability of solitary waves in the above model equations.

The one-dimensional stability of the solitary waves has been investigated so far by many authors [2–5] in the framework of the GKP equation (1). According to their studies, planar solitary waves are one-dimensionally stable if

\[ \frac{dP}{dv} > 0 , \]

where

\[ P(v) = \frac{1}{2} \int_{-\infty}^{\infty} u_s^2(x - vt; v) \, dx \]

is the momentum of the solitary wave. Here the solitary wave solutions are expressed in the form \( u = u_s(x - vt; v) \), and \( v \) is the wave velocity.

The transverse stability was first examined by Kadomtsev and Petviashvili [6]. They derived the usual Kadomtsev-Petviashvili (KP) equation which is equivalent to (2) with \( f(u) = cu^2 \) and, on the basis of this equation, studied the linear stability of planar solitary waves with respect to long-wavelength transverse perturbations. They found that the solitary waves are unstable to such perturbations in a medium with positive dispersion and stable in the case of negative dispersion. The complete linear stability analysis with no restriction on the wavelength of perturbations was later conducted by many authors [8] to show that there is a short-wavelength cutoff to the instability for the positive dispersion case. What we want to stress in their findings is, however, that for the negative dispersion case, the solitary waves are stable to any transverse perturbations within the framework of the KP equation.

In this Letter, we consider planar solitary waves which are one-dimensionally stable, and examine their linear stability to long-wavelength transverse perturbations in the framework of the GKP equation (2). We follow Kadomtsev and Petviashvili [6] to derive an evolution equation for the wave velocity that describes the slow time evolution of the solitary wave in response to the long-wavelength transverse perturbations. This equation serves as a basic equation for making the linear stability analysis. Then it is revealed that
for some nonlinearities in this family, planar solitary waves are transversely unstable even in the case of negative dispersion, which is the new finding of this Letter. In the positive dispersion case, we show that they are always unstable to long-wavelength transverse perturbations.

Let us now proceed to derive the evolution equation for the wave velocity. We assume that the GKP equation (1) has the solitary wave solutions propagating in the positive $x$ direction with a constant speed $v_0$ as $u = u_s(x - v_0t; v_0)$, and that these waves are one-dimensionally stable. When these waves are subject to perturbations with long-wavelength oscillations in the transverse ($y$) direction, they are expected to evolve slowly in time, so we may introduce the slow variables $Y$ and $T$ such that $Y = ey$ and $T = et$ where $e$ is a small parameter. Then their behavior is described by the formal asymptotic series in $e$:

$$u = u_s(\xi; v) + \alpha e u_1(\xi, Y, T) + \alpha e^2 u_2(\xi, Y, T) + \ldots,$$

where

$$\xi = x - e^{-1} \int_0^T v(Y, T') \, dT',$$

and

$$v(Y, T) = v_0 + \alpha \dot{v}(Y, T).$$

Here the effects of perturbations are included in the wave velocity and its fluctuating part scales with $\alpha$ [see (7)]. Since the purpose of this Letter is to perform the linear stability analysis, any terms nonlinear with respect to $\alpha$ are neglected.

Substituting (5) into the GKP equation (2), we obtain, at the leading order, the following equation for $u_s$ which determines the profile of the solitary wave solutions:

$$u_{s,\xi \xi} + f(u_s) - v u_s = 0,$$

where we used the boundary condition $u_s \to 0$ as $\xi \to \infty$. Here we assume that (8) admits the steady-state solitary wave solutions that satisfy $u_s \to 0$ as $\xi \to \pm \infty$. [See condition (25) in [9] or Eqs. (C.1), (C.2) in [3] which are necessary for the existence of such solutions.]

At the higher order, a set of linear inhomogeneous equations for the higher-order corrections $u_n (n = 1, 2, \ldots)$ are given, with the use of the boundary condition $u_n \to 0$ as $\xi \to \infty$, in the form

$$(L u_n)_{\xi} = F_n(u_0, u_1, \ldots, u_{n-1}),$$

where

$$u_0 = u_s(\xi; v_0),$$

and $L$ is the linear operator defined by

$$L = \frac{\partial^2}{\partial \xi^2} + f'(u_0) - v_0 \left[ f'(u) = df/du \right].$$

The inhomogeneous terms $F_n$ on the right-hand side of (9) are constituted by the lower-order solutions. It is convenient to write down the explicit form of $F_n$ for $n = 1, 2$, which are expressed as

$$F_1 = - \frac{\partial u_0}{\partial v_0} \dot{v}_T - \frac{\sigma}{2} u_0 \int_0^T \dot{v}_{yy} \, dT',$$

$$F_2 = \frac{\sigma}{2} \int_0^\xi \frac{\partial u_0}{\partial v_0} d\xi' \dot{v}_{yy} - u_1 T.$$

Now Eq. (9) has a solution which does not diverge and remains bounded as $\xi \to \pm \infty$ if the following compatibility condition is satisfied.

$$\int_{-\infty}^\infty u_0 F_n(u_0, u_1, \ldots, u_{n-1}) \, d\xi = 0,$$

where this condition is obtained by multiplying (9) by $u_0$, and then integrating with respect to $\xi$ from $-\infty$ to $\infty$. The condition (14) leads to an evolution equation for $\dot{v}(Y, T)$. When $n = 1$, we get the following leading-order evolution equation, after substitution of (12) into (14),

$$\frac{dP_0}{dv_0} \dot{v}_T + \sigma P_0 \int_0^T \dot{v}_{yy} \, dT' = 0,$$

where $P_0 = P(v_0)$ is given by (4). Equation (15) indicates that, in the case of positive dispersion ($\sigma = 1$), the one-dimensionally stable ($dP_0/dv_0 > 0$) solitary waves are unstable to perturbations with long-wavelength transverse oscillations. In the case of negative dispersion ($\sigma = -1$), however, these oscillations are harmonic at this order, so we must proceed to the next, second-order approximation to examine the stability. The second-order approximation is obtained by calculating the condition (14) at $n = 2$, and in order to get an explicit form of this approximation, we need to evaluate the first-order correction $u_1$. Here, it is convenient to express $u_1$ in the implicit form of a sum of even ($u_{1ev}$) and odd ($u_{1od}$) components,

$$u_1 = u_{1ev} + u_{1od}.$$

The governing equations for the individual components $u_{1ev}$ and $u_{1od}$ are obtained by integrating (9) at $n = 1$ using the boundary condition far ahead of the solitary wave that $u_1 \to 0$ as $\xi \to \infty$. For our purposes, only the equation for $u_{1ev}$ is presented:

$$Lu_{1ev} = G(v_0),$$

where

$$G(v) = \frac{1}{2} \left( \frac{dM}{dv} \frac{M}{2P} \frac{dP}{dv} \right) \dot{v}_T.$$
Now we turn to the second-order approximation and combine the compatibility conditions (14) for \( n = 1 \) and 2 to obtain
\[
\int_{-\infty}^{\infty} u_0(f_1 + eF_2) d\xi = 0 .
\] (21)

This condition leads to the higher-order evolution equation for \( \hat{v}(Y, T) \), which we are seeking. To calculate \( \int_{-\infty}^{\infty} u_0 u_{1T} d\xi \) that appears on the left-hand side of (21), we first note the following equation,
\[
\mathbf{L} \frac{\partial u_0}{\partial v_0} = u_0 ,
\] (22)
which is obtained by differentiating (8) with respect to \( v \), and then substituting \( v = v_0 \). Next multiplying (17) by \( \partial u_0/\partial v_0 \), and integrating with respect to \( \xi \), we find that, after using (22),
\[
\int_{-\infty}^{\infty} u_0 u_{1\xi} d\xi = \frac{dM_0}{dv_0} G(v_0) ,
\] (23)
where \( M_0 \equiv M(v_0) \). Now substitution of (23) into the compatibility condition (21) leads to the following desired evolution equation:
\[
\frac{dp_0}{dv_0} \hat{v}_T + \sigma p_0 \int_{0}^{T} \hat{v}_{YY} dT' + \frac{e \sigma p_0}{4} \left( \frac{dp_0}{dv_0} \right)^{-1} A(v_0) \hat{v}_{YY} = 0 ,
\] (24)
where
\[
A(v) = \frac{dM^2}{dv} \frac{d}{dv} \left( \log \frac{P}{|M|} \right) ,
\] (25)
and the leading-order equation (15) is used to derive (24).

The evolution equation (24) can describe the time evolution of one-dimensionally stable solitary waves satisfying both \( dp_0/dv_0 > 0 \) and \( dp_0/dv_0 = O(1) \). However, this equation is not valid for the description of the solitary waves near the one-dimensional stability threshold, or \( |dp_0/dv_0| \ll 1 \). In this case we put \( T = e^{2/3} Y = ey \) and in response to that, \( u \) is expanded as
\[
u = u_1(\xi; v) + \alpha e^{2/3} u_1(\xi, Y, T)
+ \alpha e^{4/3} u_2(\xi, Y, T) + \ldots ,
\] (26)
where
\[
\xi = x - e^{-2/3} \int_{0}^{T} v(Y, T') dT'.
\] (27)

Equations (26) and (27) are the replacements of (5) and (6). Then the inhomogeneous terms \( F_\alpha(n = 1, 2) \) of Eq. (9) for the higher-order corrections \( u_\alpha \) become
\[
F_1 = -\frac{\partial u_0}{\partial v_0} \hat{v}_T ,
\] (28)
\[
F_2 = -\frac{\sigma}{2} u_0 \int_{0}^{T} \hat{v}_{YY} dT' - u_{1T} .
\] (29)
The evolution equation for \( \hat{v}(Y, T) \) is obtained in the same manner as in the case of \( dp_0/dv_0 = O(1) \). Therefore the details are omitted and the desired evolution equation when \( |dp_0/dv_0| \ll 1 \) becomes
\[
e^{-2/3} \frac{dp_0}{dv_0} \hat{v}_T + \sigma p_0 \int_{0}^{T} \hat{v}_{YY} dT' + \frac{1}{2} \left( \frac{dM_0}{dv_0} \right)^2 \hat{v}_{TT} = 0 .
\] (30)

Note that, when the \( Y \) dependence is absent, this equation corresponds with the linearized version of the evolution equation derived by Pelinovsky and Grimshaw [see (2.18) in [21]] for describing the one-dimensional solitary wave dynamics near the stability threshold.

To make the linear stability analysis of the solitary waves, we put \( \hat{v}(Y, T) \) in the form
\[
\hat{v} = v_1 \exp(\lambda T + ilY) ,
\] (31)
where \( v_1 \) and \( \lambda \) are complex, while \( l \) is real. First, we consider the case of \( dp_0/dv_0 = O(1) \). Substituting (31) into the evolution equation (24), we obtain the following algebraic equation for the eigenvalue \( \lambda \):
\[
\lambda^2 - \sigma l^2 p_0 \left( \frac{dp_0}{dv_0} \right)^{-1} \left[ \frac{e}{4} \left( \frac{dp_0}{dv_0} \right)^{-1} A(v_0) \lambda + 1 \right] = 0 .
\] (32)
Recalling that \( e \) is small and \( dp_0/dv_0 > 0 \), (32) indicates that the one-dimensionally stable solitary waves with \( dp_0/dv_0 = O(1) \) are always unstable to long-wavelength transverse perturbations for the positive dispersion case \( \sigma = 1 \). The growth rate \( = \text{Re}[\varepsilon \lambda] \) of their perturbations is proportional to \( e \). In contrast, for the negative dispersion case \( \sigma = -1 \), we find from (32) that the solitary waves are transversely unstable if
\[
A(v_0) < 0 .
\] (33)

When the criterion (33) is satisfied, perturbations grow exponentially with time, and their growth rate is \( O(e^2) \). To know the linear stability when \( A(v_0) \to 0 \), we must proceed to the fourth-order approximation of the evolution equation whose calculation is so complicated that we could not obtain a simple criterion as (33).

When the solitary waves are near the one-dimensional stability threshold, or \( |dp_0/dv_0| \ll 1 \), the corresponding algebraic equation for the eigenvalue \( \lambda \) can be obtained, by substituting (31) into (30), as
\[
\lambda^3 + 2 \left( \frac{dM_0}{dv_0} \right)^2 \left( \frac{e^{-2/3} \mu \lambda^2 - \sigma l^2 p_0} {\mu} \right) = 0 ,
\] (34)
where \( \mu = dp_0/dv_0(|\mu| \ll 1) \) is small and real. Recalling that \( e \) is also small, and applying the method of dominant balance [10] to (34), we find the following results about the solitary wave stability: (a) When \( O(e^{2/3}) < |\mu| < O(1) \) with \( \mu > 0 \), the solitary waves are
transversely unstable for both cases of dispersion. The growth rates of perturbations \((\equiv \text{Re}[e^{2/3}])\) are \(O(\mu^{-1/2} e)\) and \(O(\mu^{-2} e^2)\) in the cases of positive and negative dispersion, respectively. (b) When \(|\mu| \leq O(e^{2/3})\), the solitary waves are again transversely unstable and the growth rate of perturbations is \(O(e^{2/3})\) for both cases of dispersion. (c) When \(O(e^{2/3}) < |\mu| < O(1)\) with \(\mu < 0\), the one-dimensional instability prevails.

We now summarize the above results concerning the stability of one-dimensionally stable solitary waves with respect to long-wavelength transverse perturbations. In the case of positive dispersion, they are always unstable. In the case of negative dispersion, their stability is determined by the criterion \((33)\) when \(dP_0/dv_0 = O(1)\), and they are unstable when \(dP_0/dv_0 \ll 1\). These are the main results of this Letter.

Let us apply the results in the negative dispersion case to the power-KdV model \([2,5]\) as a particular example. In this model the nonlinear function \(f(u)\) in the GKdV equation \((1)\) is given by \(f = (p + 2)u^{p+1}\) where \(p\) is real and positive. Then \((1)\) has the following solitary wave solutions:

\[
u_s(\xi; v) = \left[ (v/2)^{1/2} \text{sech}(p v^{1/2} \xi/2) \right]^{2/p}, \tag{35}\]

where \(\xi = x - vt\). They are known to be one-dimensionally unstable for \(p \geq 4\) \([2,5]\). The transverse stability can be obtained by an application of the above results \((33), (a),\) and \((b)\); which indicate that they are unstable to long-wavelength transverse perturbations for \(2 < p \leq 4\). (Note that, when \(p = 1\), this result is consistent with that of Kadomtsev and Petviashvili \([6]\).)

Even when the nonlinear function \(f(u)\) is more commonly given by \(f(u) = \pm(p + 2)u^{p+1} + (p + 1)u^{2p+1}\) (note that arbitrary values can be assigned to the coefficients of \(u^{p+1}\) and \(u^{2p+1}\) by a trivial scaling transformation), the solitary waves with negative dispersion are transversely unstable for \(p_{c2}(v) < p \leq p_{c1}(v)\) where \(p_{c1}(v)\) and \(p_{c2}(v)\) \((< p_{c1}(v))\) are critical values of \(p\) at the one-dimensional and the transverse stability thresholds, respectively. Therefore it is expected that, for any KdV-type solitary waves, there exists a region of transverse instability at the lower degree of nonlinearities than that of the one-dimensional instability. This fact, which is our new finding as mentioned in the introductory paragraphs, suggests that consideration of the transverse instability may be very important in examining the stability of finite-amplitude solitary waves in fluids. As far as the authors know, only the one-dimensional stability has been investigated so far for the typical solitary waves in fluids like those in water \([11]\) and in a nearly uniformly stratified fluid \([9]\). Therefore it is interesting to examine the transverse stability of these finite-amplitude solitary waves on the basis of a set of the original governing equations, which is being investigated at present.

*Electronic address: kataoka@mech.kobe-u.ac.jp