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Exposita Note

On the existence
of an implementable optimal income tax*

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Summary. This paper discusses the existence of an optimal income tax and distinguishes itself from the previous articles in two respects. In previous papers, the self selection condition was not necessarily consistent with the individual budget constraint. Furthermore, implementability in previous papers was implicit in individual ability, rather than individual income, as the basis of the tax function. We offer a different concept of the self selection conditions: Anti Normal Envy that is consistent with the individual budget constraint and that we show to be equivalent to the competitive equilibrium under a tax function based on income. We then establish the existence of an implementable optimal income tax.

Keywords and Phrases: Optimal taxation existence, Tax implementability.

JEL Classification Numbers: H21, C62, D59.

1 Introduction

Most papers concerning the theory of an optimal income tax based on the Mirrlees [7]'s model have the following characteristics:

1. Individual utility function is assumed to be strictly concave and identical for all individuals.
2. The self-selection condition is not imposed under the individual budget constraints.

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3. Importance is placed on obtaining the best mechanisms; less emphasis is placed on assuring that an optimal tax is implementable by a supporting income tax. The existence proofs are concluded based on individual ability, rather than individual income, as the basis of the optimal tax function.

This paper is also distinguished from the others by virtue of its simultaneous and consistent treatment of three main points:

i. In explicitly considering individual budget constraints, we introduce the concept of Anti Normal Envy (ANE) as a self-selection condition. Theorem 1 (the theorem of implementability) provides the necessary and sufficient conditions under which a mechanism for a tax function based on individual ability would be as implementable as that according to individual income.

ii. Theorem 3 shows that an implementable optimal income tax exists.

iii. There are infinitely many types (abilities) of individuals. In our model, the existence proof does not need the quasi-concave assumption of utility functions.

Let us explain these points in relation to the previous papers.

First, the reason why the concept of anti-normal envy plays a central role in our paper is that, within the self-selection condition, it draws a distinction between the standard theory of the principal agent problem and the optimal tax theory. If a person of lower ability can never, despite working all day without rest, achieve the higher income of a more able person, and yet still envies the more able person’s income, his envy is called strong envy. Strong envy can, in other words, arise regardless of the feasibility of the individual’s income. Neither does a selection condition that expresses the concept of Anti Strong Envy satisfy the feasibility of individual’s income either. On the other hand, normal envy is provoked when an individual who is able to achieve the income of another individual nonetheless cannot obtain higher utility than that individual’s. Hence, the Anti Normal Envy (ANE) concept stands for a selection mechanism incorporating the feasibility condition of individual income. The selection condition for the optimal tax problem must be based on ANE. The feasibility of the individual’s income is rarely made explicit in the existing literature. One of the very few to include this constraint is Berliant and Page [1], who use the term “capacity constraint”. Theorem 2 shows that under the ANE condition, there exists an optimal mechanism that is feasible and truthful.

Secondly, the existence proofs found in the literature are provided under tax functions based on individual ability rather than individual income. This is a major defect, because this kind of existence proof does not necessarily convey the implementability of the tax mechanism. To identify an optimal tax that is implementable, a tax function must be based on individual income rather than ability. To achieve this, two approaches have been taken thus far. The first approach models the optimal tax problem with a tax function that is based on income, and then shows the existence of the optimal tax. Hence, the implementability problem is naturally satisfied by its own definition. This approach is taken in Berliant and Page [1]. The second approach, which is the one adopted in the present study, extends the

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1 A mechanism is a function defined on the set of types of individuals to a commodity space.
existence proofs in the previous studies by using a supporting tax function based on income. That is, under incentive compatibility and feasibility constraints, over the class of all tax functions that are defined on ability rather than income, there exists a supporting implementable income tax mechanism to obtain the maximal social welfare. Based on this approach, our paper achieves an important result: it identifies the necessary and sufficient conditions for the implementability of a tax function based on ability (Theorem 1). There is, however, a tradeoff. Since it seems that the first approach does not need to deal with the implementability problem, its assumptions in the existence proof are more general than ours. The second approach requires some additional assumptions (compactness w.r.t. the topology of pointwise convergence and strictly increasing function of utility) in its existence proof.

Finally, in the optimal tax theory, the existence proof plays a less important role, as the first objective in this field is to obtain the necessary conditions for the optimal control problem. Moreover, the solution to the necessary conditions is a solution to the problem itself if the Hamiltonian is concave. In fact, Mirrlees [7] assumes that the utility function is strictly concave and strictly increasing. On the other hand, a few papers provided existence proofs in a different setting from that of Mirrlees. Mantel [6], Kaneko [5], and Iritani [3] gave existence proofs in a finite heterogeneous consumers’ economy with quasi-concave utility functions. It is difficult, however, to derive optimal tax properties from these models without the differentiability of functions. Very few papers have succeeded, like Berliant and Page [1], in providing an existence proof in a more-general setting without the assumption of convexity. In particular, Berliant and Page [1] also simultaneously treated the problem of the optimal provision of public goods.

2 The basic model

2.1 The economy

Consider an economy comprising of a continuum of individuals. An individual is represented by his ability to earn income and his utility. The set of abilities of all individuals is presented by \( A \) where \( A \defeq \{ a \in \mathbb{R}_+ | a \leq a \leq \pi \} \). The value of \( a \) is assumed to be positive. Individuals consume two goods: consumption \( c \) and leisure \( \ell \). Let \( X \) be the individual consumption set.\(^2\)

\[
(c, \ell) \in X \defeq [0, 2\pi] \times [0, 1].
\]  

Note that the constant value 2 in \( 2\pi \) can be any arbitrary number that is sufficiently large to be the upper bound of consumption \( c \).

Let \((A, \mathcal{B}, \mu(\cdot))\) be a measure space of individuals’ abilities where \( \mathcal{B} \) is a \( \sigma \)-algebra, and \( \mu \) which describes the population density is a probability measure.\(^3\)

We denote the utility function of an individual \( a \) by \( u(a, c, \ell) \).

\(^2\) The symbol \( \defeq \) means that the left side is defined by the right side.

\(^3\) The measure space \((A, \mathcal{B}, \mu(\cdot))\) in this paper is not necessarily the Lebesgue measure space. It implies that \( \mathcal{B} \) is not necessarily the set of Borel measurable sets, and that \( \mathcal{B} \) may contain atoms.
In general, the economy $E$ is defined as:

$$E = (u(a, (c, l))_{a \in A}, A, \mathcal{B}, \mu(\cdot)).$$

An allocation $\xi$ is a measurable function:

$$\xi : A \rightarrow X$$

$$a \mapsto (c_\xi(a), \ell_\xi(a)).$$

(2)

Given an allocation $\xi = (c_\xi(\cdot), \ell_\xi(\cdot))$, the income earned by individual $a$ is $y_\xi(a) = (1 - \ell_\xi(a)) a$. When the allocation $\xi$ is implicitly understood, the earned income can be written as $y(a) = (1 - \ell(a)) a$. From now on, we consider that an allocation $\xi$ is either $(c_\xi(a), \ell_\xi(a))$ or $(c_\xi(a), y_\xi(a))$ when the symbol $\xi$ is mentioned. This is an abuse of language, but we use it to avoid introducing other new symbols.

Let $\Xi$ be the set of allocations $\xi$ such that:

$$\Xi \overset{\text{def}}{=} \{(c_\xi(\cdot), \ell_\xi(\cdot)) \in \mathcal{B}|[c_\xi(\cdot), \ell_\xi(\cdot)) \in [0, 2\pi] \times [0, 1], \forall a \in A\},$$

(3)

where $\mathcal{B}$ is the set of measurable functions on $(A, \mathcal{B}, \mu)$.

**Assumption**

a. For each a pair $(c, \ell) \in [0, 2\pi] \times [0, 1]$, $u(a, c, \ell)$ is measurable on $A$.

b. For each $a$, $u(a, \cdot, \cdot)$ is continuous on $[0, 2\pi] \times [0, 1]$ and non-decreasing w.r.t. $c$ and $\ell$.

c. There exists some $\psi(a) : A \rightarrow \mathbb{R}$ which is integrable such that $\forall \xi \in \Xi$, $|u(a, c_\xi(a), \ell_\xi(a))| \leq |\psi(a)|$.

Note that our existence proof does not require that the utility function $u(a, \cdot, \cdot)$ to be quasi-concave.

A feasible allocation $\xi$ is an allocation which satisfies:

$$\int_A (c_\xi(a) - a(1 - \ell_\xi(a)))d\mu(a) + B \leq 0,$$

(4)

where $B \geq 0$ is the government revenue to pay for the constant government consumption.

Let $\mathcal{M}$ be the set of feasible allocations $\xi$ defined by the definition (4).

$$\mathcal{M} \overset{\text{def}}{=} \{\xi \in \Xi | \int_A (c_\xi(a) - a(1 - \ell_\xi(a)))d\mu(a) + B \leq 0\}.$$  

(5)

Denote the set of allocations $\mathcal{S}$ as follows:

$$\mathcal{S} \overset{\text{def}}{=} \{\mathfrak{S} \subset \mathcal{M} | \mathfrak{S} \text{ is sequentially compact w.r.t pointwise convergence}\}.$$  

The set $\mathcal{S}$ is not empty. In this paper, we shall work on some $\mathfrak{S} \in \mathcal{S}$.

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4 Our previous discussion paper [4] showed the examples about the non-emptiness of $\mathcal{S}$. One of them is the existence of a family of bounded variation functions due to Helly Selection Theorem (Natanson [8] page 222).

5 Note that the set $\bigcup\{\mathfrak{S}| \mathfrak{S} \in \mathcal{S}\}$ is not necessarily an element of $\mathcal{S}$. 
In this paper, revelation mechanisms will be defined as mappings from agent types into consumption and income. Hence, we define \( v(a, c, y) \) as the following:

\[
v(a, c, y) \overset{\text{def}}{=} u(a, c, 1 - \frac{y}{a}).
\]

(6)

2.2 Anti normal envy

Individual \( a \) normally envies individual \( a' \) as even though individual \( a \) is able to earn the same income as the income of individual \( a' \), his utility derived from working “truthfully” is not maximum. Mathematically:

\[
a \geq y(\xi(a')) \quad \text{and} \quad v(a, c_\xi(a'), y(\xi(a'))) > v(a, c_\xi(a), y(\xi(a))).
\]

(7) \hspace{1cm} (8)

where \( \xi \) be an allocation. An individual \( a \) is not normally envious of any other individual \( a' \) in \( A \) at the allocation \( \xi \) iff:

\[
\forall a', a \geq y(a') \Rightarrow v(a, c_\xi(a), y(\xi(a))) \geq v(a, c_\xi(a'), y(\xi(a'))).
\]

(9)

An allocation \( \xi \) is said to possess the property of anti normal envy (ANE) if it satisfies condition (9) for all \( a \) in \( A \).

Denote the set of feasible allocations satisfying ANE by \( \mathcal{F}_N \):

\[
\mathcal{F}_N \overset{\text{def}}{=} \{ \xi \in \mathcal{F} \mid \xi \text{ is an allocation satisfying ANE} \}.
\]

(10)

2.3 Implementability, ANE and anonymity

When we express an allocation \( \xi \) in \( \mathcal{F}_N \), the associating income tax function is implicit. Let it be explicit, \( T(a) \overset{\text{def}}{=} y(a) - c(a) \). Then, the consumer choice problem is described as the following:

\[
\max_{a'} v(a, y(a') - T(a'), y(a')) \quad \text{sub. to} \quad y(a') \leq a
\]

(11)

for \( a \in A \).

There remains an important problem unanswered. That is, the intrinsic choice variable in (11) is the ability \( a' \). And thus, we have to ask whether the individual choice (11) is consistent with the usual consumer choice where the tax function is based on incomes. This question leads us to an even more important problem: whether the obtained optimal tax is compatible to the general equilibrium. The problem comprises the three following sub-problems.

Problem 1 (Implementability) What should the tax be based on: individual ability or income? And then, how should we deal with the individual budget? Does the resulting allocation from the tax system maximize the individual utility under his usual budget constraint?
Problem 2 (Market Clearance) Will the demand and supply that are implemented in Problem 1 by a tax schedule based on income be in balance? The total implemented demand should not exceed the total implemented supply.

Problem 3 (Government Budget) With the total implemented demand and supply in Problem 1, is the government budget balanced? It requires that the government’s budget is not in deficit.

The meaning of Problem 1 is whether the tax bases are observable. The information on ability is private and not known to the government. Therefore, whenever tax bases are constructed from the information that is not observable by the government, their tax schedules are difficult to be institutionalized. Income tax based on individual ability in itself does not always satisfy the requirement that the income tax should be implementable.

Depending on the tax base, we classify a tax function \( T(\cdot) \) into two kinds: tax function on ability and tax function on income.

Tax function on ability \( T(a) \): Given an allocation \( \xi \equiv (c(a), y(a)) \) in the set \( \mathcal{F} \) of allocations, the associating tax function on ability \( T(a) \) is defined as:

\[
T(a) \equiv y(a) - c(a). 
\]  

(12)

Tax function on income \( T(I) \): A tax function is called tax function on income when its tax base is income.

The pair \((\xi, T)\) where the tax function \( T(a) \) on ability derived from the allocation \((c_\xi(\cdot), y_\xi(\cdot))\) is called implementable iff there exists an income tax function \( \tilde{T}(I) \) with \( I \in \mathbb{R}_+\), and satisfies these conditions:

A: \((c_\xi(\cdot), y_\xi(\cdot))\) is the solution to the utility maximization problem of the individual \( a \):

\[
\max_{c, y} v(a, c, y) \quad \text{sub. to } c = y - \tilde{T}(y(a))
\]

for all \( a \) in \( A \).

B: The resulting income tax \( \tilde{T}(y(a)) \) is the same as the tax function \( T(a) \), i.e.,

\[
T(a) = \tilde{T}(y(a)), \quad \forall a.
\]

C: The total demand should not exceed the total supply.

\[
\text{demand} = B + \int_A c_\xi(a)d\mu \leq \int_A y_\xi(a)d\mu = \text{supply}
\]

D: It requires that the government’s budget is not in deficit.

\[
\int \tilde{T}(y(a))d\mu \geq B.
\]

Conditions A and B show that the pair \((\xi, T(\cdot))\) satisfies Problem 1 above. Conditions C and D correspond to Problems 2 and 3 respectively.\(^6\)

\(^6\) Due to Walras law, either Condition C or D can be omitted. Conditions A, B, and C frames the model to be a setting of general equilibrium.
Theorem 1 (Implementability Theorem) Given an allocation \((c(\cdot), y(\cdot))\) in the set \(\mathcal{R}\) and let the corresponding tax function on ability be \(T(\cdot)\):

\((\xi, T)\) is implementable if and only if the following two conditions are satisfied:

\[ \forall a \forall a', \ y(a) = y(a') \Rightarrow c(a) = c(a') \quad \text{Anonymity Condition} \quad (13) \]

and \(\xi\) satisfies ANE.

Proof.

Sufficiency

Given an allocation \(\xi(\cdot) \overset{\text{def}}{=} (c(\cdot), y(\cdot))\) in the set \(\mathcal{R}\) satisfying (13) and ANE, define

\[ Y \overset{\text{def}}{=} \{ y(a) \in \mathbb{R}_+ \mid a \in A \}. \]

Define the tax function on income \(\tilde{T}(I)\) derived from the allocation \((c(\cdot), y(\cdot))\) as:

\[ \tilde{T}(I) \overset{\text{def}}{=} \begin{cases} y(a) - c(a) & \text{if } I \in Y, I = y(a) \\ I + 1 & \text{if } I \notin Y. \end{cases} \quad (14) \]

Due to (13), the tax function on income \(\tilde{T}(I)\) is well-defined.

The utility maximization problem of the individual \(a\) is described as:

\[ \max_{(c,y) \in [0,2a] \times [0,a]} v(a, c, y) \quad \text{sub. to } c = y - \tilde{T}(y). \quad (15) \]

Suppose that for some \((c^*, y^*) \in [0,2a] \times [0,a]\) with \(c^* = y^* - \tilde{T}(y^*)\), the following relation:

\[ v(a, c^*, y^*) > v(a, c(a), y(a)) \quad (16) \]

holds. Since \((c^*, y^*) \in [0,2a] \times [0,a]\), then \(y^* \in Y\). In order to see it, assume that \(y^*\) does not belong to \(Y\), then \(y^* - \tilde{T}(y^*) = -1 < 0\). Therefore, \(c^*\) does not belong to \([0,2a]\). This is a contradiction. Consequently, there exists some \(\hat{a}\) such that \(y^* = y(\hat{a})\). This leads us to

\[ c^* = y^* - \tilde{T}(y^*) = y(\hat{a}) - \tilde{T}(y(\hat{a})) = y(\hat{a}) - (y(\hat{a}) - c(\hat{a})) = c(\hat{a}). \]

As a result,

\[ a \geq y(\hat{a}) \text{ and } v(a, c(\hat{a}), y(\hat{a})) > v(a, c(a), y(a)). \]

Owing to ANE condition, the set of \(a\) that satisfies (16) is an empty set. Therefore, the pair \((c(a), y(a))\) is the solution to the problem (15) for all \(a \in A\).

In addition, the following relation holds due to (13) and (14)

\[ T(a) = \tilde{T}(y(a)) \text{ for all } a \in A. \]
Finally, the requirement on government’s budget is satisfied, since

\[
\text{Net Government Revenue} = \int \hat{T}(y(a))d\mu
\]

\[
= \int ((y(a)) - c(a))d\mu \geq B.
\]

It is also obvious that the total demand does not exceed the total supply by the feasibility of \( \xi \).

**Necessity** Self-evident.

3 Existence theorem

In this section, there are two theorems. Theorem 2 shows that there exists the optimal tax satisfying ANE under the assumption of the sequential compactness w.r.t. pointwise convergence. Theorem 3 concludes the existence of the optimal implementable tax under the assumption of strictly increasing utility function.

Define a preorder on the set \( \bar{F} \subset F \) by \( \preceq_{\alpha} \) which is expressed as follows:

\[
\forall \xi, \xi' \in \bar{F}, \quad \xi \preceq_{\alpha} \xi' \iff \int_{A} \alpha(a)v(a, c_\xi(a), y_\xi(a))d\mu \geq \int_{A} \alpha(a)v(a, c_{\xi'}(a), y_{\xi'}(a))d\mu,
\]

(17)

where \( \alpha(\cdot) \in L^\infty(A, \mathfrak{B}, \mu) \) and \( \alpha : A \to \mathbb{R}_+ \) is a function. The value \( \alpha(a) \) can be considered as some social weight, that is, the relative social importance of individual \( a \). The relation \( \preceq_{\alpha} \) is obviously a complete preorder in \( \bar{F} \).

We will show that there exists a maximal element with respect to \( \preceq_{\alpha} \) when \( \bar{F} = \bar{F}_N \), that is, \( \xi \) and \( \xi' \) are chosen from \( \bar{F}_N \).

3.1 Existence theorem

**Theorem 2 (Existence Theorem under Anti Normal Envy)** Under the assumption \( a, b, c \) and that \( \bar{F}_N \) is not empty, there exists a maximal allocation with respect to the preorder \( \preceq_{\alpha} \) on the set \( \bar{F}_N \) of all feasible allocations satisfying ANE.

**Proof.** It will suffice to show that (i) the continuity of \( \int \alpha(a)v(a, \xi(a))d\mu \) w.r.t. \( \xi \in \bar{F}_N \), and that (ii) \( \bar{F}_N \) is sequentially compact w.r.t. pointwise convergence topology.

Suppose \( \xi^\nu \in \bar{F}_N, \nu = 1, 2, \ldots \) converges pointwise to a \( \xi^* \) in \( \bar{F}_N \). Then \( v(\cdot, \xi^\nu(\cdot)) \) converges pointwise to \( v(\cdot, \xi^*(\cdot)) \) because of Assumption b.

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7 We do not require the convex preferences in this theorem. Our previous discussion paper [4] showed an example illustrating an allocation satisfying ANE for a non quasi-concave utility function \( v(a, c, y) \) which meets all the above mentioned assumptions.
Moreover, due to Assumption c, there exist a function $\psi(a) : A \rightarrow \mathbb{R}$ such that $|v(a, \xi^*(a))| \leq |\psi(a)|$ and $|v(a, \xi^\nu(a))| \leq |\psi(a)|$. Hence, $v(a, \xi^*(a))$ and $v(a, \xi^\nu(a))$ are integrable functions (Halmos [2], page 112).

From bounded convergence theorem, we obtain the continuity:

$$\int \alpha(a)v(a, \xi^\nu(a))d\mu \rightarrow \int \alpha(a)v(a, \xi^*(a))d\mu. \quad (18)$$

Let $\{\xi^\nu\}_{\nu=1}^\infty$ be a sequence in $\mathcal{F}_N \subset \mathcal{F}$, where $\xi^\nu \overset{\text{def}}{=} (c^\nu(\cdot), y^\nu(\cdot))$. Since $\mathcal{F}$ is sequentially compact, there exists a subsequence converging to $\xi^* \overset{\text{def}}{=} (c^*(\cdot), y^*(\cdot))$ in $\mathcal{F}$. Let the subsequence be $\{\xi^\nu\}_{\nu=1}^\infty$ itself.

We need to show that $\xi^* \in \mathcal{F}_N$. For any $a \in [a, \bar{a}]$, it holds that as $\nu' \rightarrow \infty$,

$$\xi^\nu' (a) \rightarrow \xi^*(a).$$

It is easy to see that $\xi^* \in \mathcal{F}$ (apply the Dominated Convergence Theorem). Now, we need to show that $\xi^*(\cdot) \in \mathcal{F}_N$. Suppose not. Then for some $a \in A$ and $a' \in A$, with $y(a') \leq a$,

$$v(a, c^*(a'), y^*(a')) > v(a, c^*(a), y^*(a')). \quad (19)$$

We already knew

$$c^\nu(a') \rightarrow c^*(a') \quad y^\nu(a') \rightarrow y^*(a')$$

$$c^\nu(a) \rightarrow c^*(a) \quad y^\nu(a) \rightarrow y^*(a).$$

Due to the continuity of the utility function $v$, there is a number $\overline{\nu}$ such that for any $\nu$ satisfying $\nu \geq \overline{\nu}$:

$$v(a, c^\nu(a'), y^\nu(a')) > v(a, c^\nu(a), y^\nu(a')). \quad (20)$$

This contradicts the fact that $\xi^\nu$ satisfies ANE. Thus $\xi^*(\cdot) \in \mathcal{F}_N$.

Because $\int \alpha(a)v(a, \xi(a))d\mu$ is continuous w.r.t. $\xi \in \mathcal{M}$ and $\mathcal{F}_N$ is sequentially compact w.r.t. pointwise convergence topology, there is a maximal allocation over $\mathcal{F}_N$. This accomplishes the proof. \hfill \Box

The above theorem does not assure that the obtained optimal tax is implementable. In order to show this point, we need an extra Assumption d on the individual utility:

**Assumption d**

For each $a$, $u(a, \cdot, \cdot)$ is a strictly increasing function w.r.t. $c$ and $\ell$.

**Lemma 1** Under Assumption d, ANE implies the Anonymity Condition.
Proof. For an allocation $\xi$ in $\mathcal{F}_N$, let $y(a) = y(a') \leq \min(a, a')$. Due to ANE condition,

$$v(a, c_\xi(a), y_\xi(a)) \geq v(a, c_\xi(a'), y_\xi(a')).$$

Then, from the assumption $d$, $c_\xi(a) \geq c_\xi(a')$. Also due to ANE,

$$v(a', c_\xi(a'), y_\xi(a')) \geq v(a', c_\xi(a), y_\xi(a)).$$

Hence, $c_\xi(a') \leq c_\xi(a)$. Consequently, $c_\xi(a) = c_\xi(a')$.

With Theorem 1 and Lemma 1, the optimal allocation in Theorem 2 is implementable. Theorem 3 concludes this result.

**Theorem 3 (Existence Theorem of Optimal Implementable Tax) Under Assumptions $a$, $b$, $c$ and $d$, there is an implementable optimal income tax in $\mathcal{F}_N$.**

\[\square\]

**References**