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Stability Analysis of Hybrid Automata with Set-Valued Vector Fields Using Sums of Squares

Izumi MASUBUCHI, Member and Tokihisa TSUJI, Nonmember

Summary

Stability analysis is one of the most important problems in analysis of hybrid dynamical systems. In this paper, a computational method of Lyapunov functions is proposed for stability analysis of hybrid automata that have set-valued vector fields. For this purpose, a formulation of matrix-valued sums of squares is provided and applied to derive an LMI/LME problem whose solution yields a Lyapunov function.

Key words: hybrid automata, robust stability, L2-gain, sums of squares, LMIs

1. Introduction

One of the most important notions in theory of dynamical systems is stability, under which the variables of systems evolve in an adequate region. Though it is the most basic property of dynamical systems, analysis of stability is not very easy except for such linear systems. A considerable amount of attention has been paid to stability analysis of hybrid dynamical systems (HDS) in the last decade. Various extensions of notions of stability for differential equations have been proposed for HDS [3], [9]–[11], where the theory of Lyapunov has been generalized to deal with executions of HDS, which consist of continuous and discrete evolutions. Computational methods have been shown for piecewise linear systems [4], [9] and more general classes of HDS [5], [7], where the latter utilize sums of squares [6], [8] with which semidefinite programming is applicable to problems of seeking positive functions.

In this paper, extending the results of [5], we propose a computational method of Lyapunov functions for stability analysis of hybrid automata [1], [2], [5], [12] that have set-valued vector fields for continuous evolution. Admitting set-valued vector fields enables to handle more general classes of HDS than those ever considered before such as in [5], [7]. Then stability analysis involves what corresponds to robustness analysis in continuous dynamical systems. We reduce a Lyapunov-like stability condition to a condition of matrix inequalities of functions and apply matrix-valued sums of squares to solve those inequalities. For this purpose we generalize the formulation of scalar-valued sums of squares [6] to matrix-valued and show linear matrix inequalities and equations (LMIs and LMEs in short) whose solution yields a Lyapunov function of HDS. LMIs are convex inequalities and solving LMIs/LMEs enjoys fast and globally convergent numerical algorithms.

Notation. Denote by $\mathbb{R}^p$, $\mathbb{R}^{p\times r}$ and $\mathbb{R}^{p\times q\times r}$ the sets of vectors, matrices and (3,0)-tensors of real numbers, respectively. Elements of $v \in \mathbb{R}^p$, $M \in \mathbb{R}^{p\times r}$, $T \in \mathbb{R}^{p\times q\times r}$ are represented as $v^i$, $M_{ik}$, $T_{ijk}$ respectively. The elements of the tensor product $T = v \otimes M$ are $T_{ijk} = v^i M_{jk}$. Let $M^*$ and $M^t$ stand for the i-th row vector and the j-th column vector of matrix $M$, respectively. We define inner products for $\mathbb{R}^p$, $\mathbb{R}^{p\times q}$, $\mathbb{R}^{p\times q\times r}$ by

$$\langle v, w \rangle = \sum_i v^i w^i, \quad v, w \in \mathbb{R}^p,$$

$$\langle M, N \rangle = \sum_{jk} M_{jk} N_{jk}, \quad M, N \in \mathbb{R}^{p\times r},$$

$$\langle T, U \rangle = \sum_{i,j,k} T_{ijk} U_{ijk}, \quad T, U \in \mathbb{R}^{p\times q\times r}.$$

For $v \in \mathbb{R}^p$ and $T \in \mathbb{R}^{p\times q\times r}$, we denote by $\langle v, T \rangle$ the contraction of $v \in \mathbb{R}^p$ and $T \in \mathbb{R}^{p\times q\times r}$, where $\langle v, T \rangle$ is a $q \times r$ matrix whose $(j,k)$-element is given as $\langle v, T \rangle_{jk} = \sum_i v^i T_{ijk}$. Let $F(\cdot)$ be a linear map from $\mathbb{R}^{p\times q\times r}$ to $\mathbb{R}^{q\times r}$. Then the adjoint of $F(\cdot)$ is defined as the map $F'(\cdot)$ that satisfies

$$\langle F(T), Z \rangle = \langle T, F'(Z) \rangle$$

for all $T \in \mathbb{R}^{p\times q\times r}$ and $Z \in \mathbb{R}^{q\times r}$. Let $M^T$ denote the transpose of a matrix (or vector) $M$. For a symmetric matrix $M$ the inequality $M > (\geq) 0$ means that $M$ is positive (semi-)definite. The standard Euclidean norm is denoted by $\| \cdot \|$.  

2. Preliminaries

2.1 Hybrid Automata and Executions

Let us define a class of hybrid automata and their executions based on [12]. For a finite collection $V$ of variables $v$, let $V$ (in boldface) denote the set of valuations of the variables. Variables whose set of valuation is finite or countable are referred to as discrete and to variables whose set of valuations is a subset of a Euclidean space or a manifold as continuous.

Definition 1: A hybrid automaton $HA$ is a collection...
Example 1: Let us consider a hybrid automaton of the class of Definition 1 with:

- \( Q \) is a finite collection of discrete variables with \( Q \) being a finite set. An element of \( Q \) is referred to as a mode;
- \( X \) is a finite collection of continuous variables with \( X = R^n \) for \( n \) every \( q \) in \( Q \);
- \( \text{Init} \subset \Xi \) is a set of initial values, where \( \Xi = \{(q, x) : q \in Q \} \);
- \( F : (q, x) \in \Xi \to F_q(x) \in 2^\mathbb{T}X \) is a set-valued vector field;
- \( D : q \in Q \to D_q \in 2^\mathbb{R}^n \) is a map assigning a subset of \( \mathbb{R}^n \) to each \( q \) in \( Q \);
- \( E \subset Q \times \mathbb{Q} \) is a set of edges, where \( e = (r, q) \in E \) means that the graph of the automaton has an edge from \( r \) to \( q \);
- \( G : e = (r, q) \in E \to G_e \in \mathbb{R}^n \) is a map assigning a subset of \( \mathbb{R}^n \) to each edge \( e \);
- \( R : (e, x) \in E \times X \to R_e(x) \in 2^\mathbb{R}^n \) with \( e = (r, q) \) is a map called reset that assigns a subset of \( \mathbb{R}^n \) to each \( e = (r, q) \in E \) and \( x \in G_e \).

Then the hybrid automaton \( \text{HA} \) represents a hybrid dynamical system with two modes \('+', '-'\) and the continuous variable in \( \mathbb{R}^2 \). The continuous variable \( x \) in \( \mathbb{R}^2 \) with \( x \) belonging to the set \( F_q(x) \) as far as \( x \) remains in the mode invariant set \( D_q \). Suppose \( q = ' + ' \). If \( x \) arrives at a point in \( G_{e_1} \), i.e., if \( x(t) = x_1 \), then the mode changes to \(' - '\) where \( e_1 = (-, +) \) corresponds to mode transition from \(' + ' \) to \(' - ' \). The edge \( E \) is the set of all possible transitions of the modes and thus the mode can change also from \(' - ' \) to \(' + ' \). According to Definition 1, the continuous variable \( x \) does not jump at mode transitions.

The descriptions of the evolution of the variable \( q, x \) in Example 1 are formalized below by defining hybrid time trajectory and executions, which follow those in [12]. The hybrid time trajectory is a sequence of time intervals in which the discrete variable is constant. The execution is defined as a function of a hybrid time trajectory that is accepted by the hybrid automaton \( \text{HA} \).

\[ F_q(x_q) = \{0\}, q \in Q_0, \]

\[ e = (r, q) \in E \quad \text{and} \quad x \in Q_0 \Rightarrow q \in Q_0 \]

Below we assume \( x_q = 0 \) for all \( q \in Q_0 \) without loss of generality. Then \( \text{Equi} = \{(q, 0) : q \in Q_0 \} \). We define a notion of stability of the continuous variable as a generalization of Lyapunov stability of continuous dynamical systems.

Definition 4: Let \( Q_0 \) be a subset of \( Q \). A set \( \text{Equi} = \{(q, x_q) : q \in Q_0, x_q \in D_q \} \) is an equilibrium set if:

- \( F_q(x_q) = \{0\}, q \in Q_0 \);
- \( e = (r, q) \in E \quad \text{and} \quad x \in Q_0 \Rightarrow q \in Q_0 \)

Definition 5: Let \( \text{Equi} \) be defined above. The set \( \text{Equi} \) is said to be stable if for any \( e > 0 \) there exists \( \delta > 0 \) such that:

\[ ||x_0(t_0)|| < \delta \Rightarrow ||x(t)|| < \delta, \quad \forall k \in \{1, ..., N\}, \quad \forall t \in I_k \]

holds for every execution \( \chi = \{I_k, q_k, x_k\} \) accepted by \( \text{HA} \).

Based on [9], [10], we use a Lyapunov-like stability condition of executions shown below. Denote by \( R^* \) the set of nonnegative real numbers. A continuous function \( \alpha : R^* \to R^* \) is said to be a class-\( \mathcal{K} \) function if \( \alpha(0) = 0 \)

\[ \alpha(r_1) < \alpha(r_2) \quad \text{for any} \quad 0 < r_1 < r_2. \]

\( 7X \) is the tangent space of \( X \). In this paper we can always identify it as \( \mathbb{R}^n \).

We sometimes use notation such as \( D_q, F_q(x) \) to represent maps with discrete variables such as \( q \).
Proposition 1: The set Equi is stable if there exist $V_q(x):(q,x) \in \Xi \to \mathbb{R}$, where $V_q$ is $C^1$ for fixed $q$, and class-$\mathcal{K}$ functions $\alpha(\cdot), \beta(\cdot)$ for which the following conditions hold:

\[
\alpha(||x||) \leq V_q(x) \leq \beta(||x||), \quad \forall q \in \mathbb{Q}, \forall x \in D_q,
\]

(3)

\[
\frac{\partial V_q(x)}{\partial x} v \leq 0, \quad \forall v \in F_q(x), \forall q \in \mathbb{Q}, \forall x \in D_q,
\]

(4)

\[
V_q(R_q(x)) \leq V_q(x), \quad \forall e = (r,q) \in E, \forall x \in G_e.
\]

(5)

We call $V_q(x)$ satisfying the conditions of the above proposition Lyapunov function.

Example 2: Consider the hybrid automaton given in Example 1. Let $Q_0 = \mathbb{Q}$. Then Equi = $\{(q,0) : q \in Q_0\}$ is stable as seen below. Set $V_q(x) = (x_1^2 + x_2^2)/2$. Then the conditions (3) and (5) are obvious. For $v = F_3(x)$,

\[
\frac{\partial V_q(x)}{\partial x} v = -3x_1^2 + x_1^2 - x_2^2 + \Delta(x_1^2 + x_1x_2)
\]

(6)

for some $\Delta \in [-1,1]$, which implies

\[
6 \leq -3x_1^2 + x_1^2 - x_2^2 + (x_1^2 + |x_1x_2|)
\]

\[
= -\frac{7}{4}x_1^2 + \left(1 \pm \frac{4}{7} x_1\right) - \frac{1}{2}|x_1| - |x_2|^2
\]

(7)

From the definition of $D_q$, $1 + 4x_1/7 \geq 3/7$ if $q = +$ since $x \in D$, implies $x_1 \geq -1$. The same holds also for $q = -$.

Therefore (7) is nonpositive.

\[
\square
\]

2.3 Matrix-Valued Sums of Squares

In Sect. 3, we consider stability analysis of a class of hybrid automata via numerical methods. This subsection is devoted to formulate matrix-valued sums of squares as a generalization of (scalar-valued) sums of squares shown in [6]. Sums of squares have been widely used to derive positive functions, such as Lyapunov functions (e.g., [5], [7]). We utilize matrix-valued sums of squares to compute positive semidefinite matrices that are functions of $(x,q)$, by which stability analysis is solved via convex optimization.

Let $\omega^{(j)}(x): x \in \mathbb{R}^n \to \mathbb{R}^p$, $m = 1, \ldots, M$ be linearly independent vector-valued functions. Denote the set of these functions by

\[
S = \{\omega^{(1)}(x), \ldots, \omega^{(M)}(x)\}
\]

and the span of $S$ by

\[
\mathcal{F}(S) = \text{span } [S]
\]

\[
= \left\{ s(x) = \sum_{m=1}^M c_m \omega^{(m)}(x) : c_1, \ldots, c_M \in \mathbb{R} \right\}
\]

Let $\mathcal{K}(S)$ be the set of finite linear combinations of dyads of elements of $S$:

\[
\mathcal{K}(S) = \left\{ \sum_{l=1}^L s^{(l)}(x)s^{(l)}(x)^T : s^{(1)}, \ldots, s^{(k)} \in S \right\}
\]

where the number of summand $K_0$ depends on $W(x)$. This is a subset of $p \times p$-matrix-valued functions and we refer to elements of $\mathcal{K}(S)$ as matrix-valued sums of squares. Denote $\Omega(x) = \left[ \omega^{(1)}(x) \cdots \omega^{(M)}(x) \right]$ and define a vector-valued function $l(x): x \in \mathbb{R}^n \to \mathbb{R}^N$ whose elements $l_i(x)$, $i = 1, \ldots, N$ are linearly independent and a linear map $\Lambda(T): T \in \mathbb{R}^{N \times p} \to \mathbb{R}^{M \times M}$ that satisfy

\[
\Lambda(l(x) \otimes R) = \Omega(x)^T R \Omega(x)
\]

(8)

for all $x \in \mathbb{R}^n$ and $R \in \mathbb{R}^{p \times p}$.

Remark 1: The map $\Lambda(\cdot)$ is (8) is a generalization of that defined in [6] and they coincide with each other for $p = 1$. Since $\Lambda(\cdot)$ is linear, it is represented in terms of elements as $[\Lambda(T)]^{mn} = \sum_{j,k} \Lambda^{mn}_{jk} T^{jk}$, where $\Lambda^{mn}_{jk}$'s are the coefficients of the linear map $\Lambda(\cdot)$. Then (8) is equivalent to

\[
\sum_{i} \Lambda^{mn}_{jk} [l_i(x)]^2 = \omega^{(m)}(x) \omega^{(n)}(x).
\]

(9)

The following lemma gives a representation of $\mathcal{K}(S)$.

Lemma 1: A matrix-valued function $W(x)$ belongs to $\mathcal{K}(S)$ if and only if it is represented with a positive semidefinite matrix $Y \in \mathbb{R}^{M \times M}$ as

\[
W(x) = \Omega(x)^T Y \Omega(x)^T = \langle l(x), \Lambda^*(Y) \rangle.
\]

(10)

Proof. Suppose that $W(x) \in \mathcal{K}(S)$. Then for some $s^{(l)} = \sum_{i,j,k} c_{ijk} \omega^{(j)}(x) \omega^{(k)}(x)^T$ we have

\[
W(x) = \sum_{i,j,k} K_{ij} \omega^{(j)}(x) \omega^{(k)}(x)^T = \sum_{i,j,k} M_{ij,k} \omega^{(j)}(x) \omega^{(k)}(x)^T.
\]

Define $Y_{jk}^{(i)} = \sum_{i,j,k} K_{ij} c_{ijk}$. Then $Y \geq 0$ and

\[
W(x) = \sum_{i,j,k} Y^{(i)}_{jk} \omega^{(j)}(x) \omega^{(k)}(x)^T = \Omega(x)^T Y \Omega(x)^T.
\]

(10)

Following the above conversely, we see $W(x) \in \mathcal{K}(S)$ if $W(x) = \Omega(x)^T Y \Omega(x)$, where $c_{ijk}$'s are defined as the elements of $C$ that decomposes $Y \geq 0$ as $Y = CC^T$.

The second equality in (9) is shown as follows. For any $Y \in \mathbb{R}^{M \times M}$ and $R \in \mathbb{R}^{p \times p}$, we have

\[
\langle \Lambda(l(x) \otimes R), Y \rangle = \langle l(x) \otimes R, \Lambda^*(Y) \rangle
\]

\[
= \sum_{i,j,k} [l_i(x)]^T R_{jk} \sum_{m,n} \Lambda^{mn}_{jk} \omega^{(m)}(x) \omega^{(n)}(x)^T
\]

\[
= \sum_{i,j,k} R_{jk} \left( \sum_{l} [l_i(x)]^T \sum_{i,j,k} \Lambda^{mn}_{jk} \omega^{(m)}(x) \omega^{(n)}(x)^T \right)
\]

\[
= \langle R, \langle l(x), \Lambda^*(Y) \rangle \rangle,
\]

(10)
while (8) and the first equality of (9) imply
\[ \langle A(l(x) \otimes R), Y \rangle = \langle \Omega(x)R\Omega(x)^T, Y \rangle \]
\[ = Tr\Omega(x)R^T\Omega(x)^TY = TrR^T\Omega(x)^TY\Omega(x) \]
\[ = \langle R, \Omega(x)^T\Omega(x) \rangle = \langle R, W(x) \rangle . \]
(11)
Since R is arbitrary, we obtain W(x) = \langle l(x), \Lambda^*(Y) \rangle from (10) and (11).
\[ \square \]

Remark 2: The representation of (9) in terms of l(x) is a generalization of [6] for scalar-valued sums of squares to matrix-valued. Below in this paper we utilize this expression.
\[ \square \]

Remark 3: The element of the matrix W(x) in (9) is [W(x)]^k = \sum_{m,n} A_{ijk} Y_{mn}[l(x)]^j, which is N-vector valued as l(x), is given by
\[ [\Lambda^*(Y)]^j = \begin{bmatrix} \sum_{m,n} A_{i}^{mn} Y_{mn} \\ \vdots \\ \sum_{m,n} A_{n}^{ij} Y_{mn} \end{bmatrix} . \]
Thus we have [W(x)]^k = \langle l(x), [\Lambda^*(Y)]^j \rangle, where \langle \bullet, \bullet \rangle is the interior product of N-vectors.
\[ \square \]

3. Stability Analysis via Matrix-Valued Sums of Squares

3.1 Description of a Class of Hybrid Automata

We consider hybrid automata that have the following expressions, which we utilize to carry out stability analysis via a computational method based on matrix-valued sums of squares, as well as we make several assumptions.

1. The map F_q(x), which assigns a set of vector field to (q, x), is represented as
\[ F_q(x) = \{ f_q^A(x) + f_q^B(x)\Delta f_q^C(x) : \Delta \leq k_q \} , \]
where f_q^A(x) \in \mathbb{R}^{n_q}, f_q^B(x) \in \mathbb{R}^{n_q \times n_q} and f_q^C(x) \in \mathbb{R}^{n_q \times n_q} with \mathbb{R}^n being identified as \mathbb{R}^{n_q}. The scalar k_q \geq 0 gives the upper bound of the largest singular value, denoted by \Delta, of \Delta \in \mathbb{R}^{n_q \times n_q}. For each execution, x(t) satisfies the following differential equation
\[ \dot{x}_q(t) = f_q^A(x(t)) + f_q^B(x(t))\Delta f_q^C(x(t)) \]
(13)
for some \Delta_q(t, x) satisfying \partial(\Delta_q(t, x)) \leq k_q.

2. The mode-invariant sets are given as D_q = \{ x : d_q(x) \geq 0, i = 1, \ldots, N_{D_q} \}, where d_q : \mathbb{R}^{n_q} \rightarrow \mathbb{R}, i = 1, \ldots, N_{D_q} are smooth functions on \mathbb{R}^{n_q}. 

3. The reset map is a singleton: R_q(x) = \{ R_q^0(x) \} with R_q^0(x); (e, x) \in E \times G_e \rightarrow \mathbb{R}^{n_e} with e = (r, q).

4. The guard G_q, e = (r, q) and the mode-invariant set satisfy \bigcup_j G_{r(e,q)} = \partial D_q.

5. The guard for e = (r, q) is represented as G_q = \bigcup_{i=1}^{N_{G_q}} \{ q^0_i(x) : x \in G_{r(e,q)} \} with g^0_i(x) = \{ x \in \mathbb{R}^{n_q} : g^0_i(x) \geq 0, i = 1, \ldots, N_{G_q} \}, where g^0_i : \mathbb{R}^{n_q} \rightarrow \mathbb{R}_{+}, i = 1, \ldots, N_{G_q} are smooth functions on \mathbb{R}^{n_q}.

6. Init \subset \{ (q, x) : x \in D_q, q \in Q \}.

The explicit expressions of F_q(x), D_q, R_q(x) and G_q are given in the items 1, 2, 3 and 5, respectively, while the assumptions in the items 1, 4, 6 ensure the existence of executions [5].

3.2 Stability Condition

We invoke a widely-known technique to handle set-valued vector fields parameterized with norm bounded elements as
\[ \alpha(||x||) \leq V_q(x) \leq \beta(||x||), \quad \forall x \in D_q, \quad q \in Q . \]
(14)
\[ \frac{dV_q(x)}{dt} - Q_q^A(x) + \sigma_q(x)(f_q^A(x))^T f_q^A(x) \]
\[ + \frac{1}{4\sigma_q(x)} \frac{dV_q(x)}{dt} f_q^B(x) f_q^B(x)^T \frac{dV_q(x)}{dt} \leq 0 , \]
\[ \quad x \in D_q, \quad q \in Q . \]
(15)
\[ V_q(R_q^0(x)) \leq V_q(x), \quad x \in G_q, \quad e = (r, q) \in E . \]
(16)
\[ \square \]

Example 3: The hybrid automaton in Example 1 can be represented as (12) with
\[ f_q^A(x) = \begin{bmatrix} -3x_1 + x_2^2 \\ -x_2 \end{bmatrix} , \quad f_q^B(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \quad f_q^C(x) = x_1 , \quad k_q = 1 . \]
This hybrid automaton satisfies (14)–(16) with V_q(x) = (x_1^2 + x_2^2)/2 and \sigma_q(x) = 1. In fact, it holds that
\[ (\text{the left-hand side of (15)}) \]
\[ = \frac{5}{3} x_1^2 \left( 1 \pm \frac{3}{5} x_1 \right) - \frac{3}{4} \left( \frac{1}{3} x_1 - x_2 \right)^2 \]
\[ \leq \frac{-2}{3} x_1^2 - \frac{3}{4} \left( \frac{1}{3} x_1 - x_2 \right)^2 \leq 0 , \]
\[ \text{since } 1 \pm (3/5)x_1 \geq 2/5 \text{ due to } x \in D_q . \]
\[ \square \]

The proof of Lemma 2 follows easily as that of quadratic stability for linear uncertain systems, by observing that the inequalities (14)–(16) imply (3)–(5). We focus on obtaining functions V_q(x) and \sigma_q(x) that satisfy the conditions of Lemma 2, based on matrix-valued sums of squares.
in Sect. 2.3. First, in general, a symmetric-matrix-valued function $A(x)$ is guaranteed to be positive on a given region $X_g := \{ x \in \mathbb{R}^n : g_i(x) \geq 0, i = 1, \ldots, k \}$ whenever there exist functions $\lambda_i(x)$ satisfying

$$A(x) - \sum_{i=1}^{k} \lambda_i(x) g_i(x) \geq 0, \quad \lambda_i(x) \geq 0, \quad i = 1, \ldots, k$$

(17)

for all $x \in \mathbb{R}^n$. Now let $\lambda_i(x)$'s and the left-hand side of the first inequality of (17) be sums of squares. Then with appropriate basis matrices $\Omega_k(x)$ and positive semidefinite matrices $Y_i$ for $i = 0, 1, \ldots, k$ the inequality condition (17) is rewritten as

$$A(x) - \sum_{i=1}^{k} g_i(x) \Omega_k(x) Y_i \Omega_k(x) = \Omega_0(x) Y_0 \Omega_0(x),$$

which has the following alternative representation

$$A(x) = \Omega(x)^T Y \Omega(x),$$

where $Y = \text{diag}(Y_0, Y_1, \ldots, Y_k)$ and

$$\Omega(x) = \begin{bmatrix} \Omega_0(x) \\ \sqrt{g_1(x)} \Omega_1(x) \\ \vdots \\ \sqrt{g_k(x)} \Omega_k(x) \end{bmatrix}, \quad x \in X_g.$$

From these observations, we can utilize positive functions that are defined and positive only in such as involving terms of $\sqrt{d_{qj}(x)}$ and $\sqrt{d_{qr}(x)}$ from the expressions of the mode-invariant sets and the guards, respectively.

Now we consider sum-of-squares formulation to obtain a Lyapunov function $V_q(x)$. Choose basis $\omega^{(m)}_q(x) : x \in D_q \rightarrow \mathbb{R}, \ m = 1, \ldots, M_q, \ q \in Q$ defined on $D_q$ and derive $l_q(x)$ and $\Lambda_q(x)$ as the counter parts of $l(x)$ and $\Lambda(x)$ in Sect. 2.3, respectively. Then we set

$$V_q(x) = \langle l_q(x), \Lambda_q^T(x) \rangle, \quad Y_q = Y_q^T \in \mathbb{R}^{M_q \times M_q}.$$

In the same way, let

$$\sigma_q(x) = \langle l_q(x), \Lambda_q^T(x) \rangle, \quad Y_q = Y_q^T \in \mathbb{R}^{M_q \times M_q},$$

where $l_q(x)$ and $\Lambda_q(x)$ are generated from basis $\omega_q^{(m)}(x)$: $x \in D_q \rightarrow \mathbb{R}, \ m = 1, \ldots, M_q, \ q \in Q$. These parameterizations of $V_q(x)$ and $\sigma_q(x)$ imply their nonnegativity on $D_q$ if $Y_q \succeq 0, \ Y_q \succeq 0$. One can easily modify the development of this subsection to strictly guarantee (14) and $\sigma_q(x) > 0$; the detail is omitted.

Next, consider the condition (15). Schur complement yields the following equivalent matrix inequality:

$$\begin{bmatrix} \frac{\partial V_q(x)}{\partial x} A_q(x) + \sigma_q(x)[A_q^T(x)]^2 \\ 1 - \sigma_q(x) l_q(x) \end{bmatrix} =: W_q(x) \leq 0,$$

(18)

where the left-hand side is symmetric and * represents off-diagonal elements abbreviated. Choose basis $\omega^{(m)}_q(x) : x \in D_q \rightarrow \mathbb{R}^{n+1}, \ m = 1, \ldots, M_q, \ q \in Q$ and derive $l_q(x)$ and $\Lambda_q(x)$ as in Sect. 2.3. Then consider the following equation

$$W_q(x) = -\langle l_q(x), \Lambda_q^T(x) \rangle,$$

(19)

which implies (15), where $\bar{Y}_q = \bar{Y}_q^T \in \mathbb{R}^{M_q \times M_q}$. We derive an equivalent equation to (19) that does not involve the variable $x$. Define matrices $\mathcal{A}_q, \mathcal{B}_q, \mathcal{C}_q, \mathcal{D}_q, \mathcal{E}_q$ with appropriate sizes so that for some vector-valued function $\phi(x), x \in D_q, \ q \in Q$, whose elements are linearly independent to each other, for which the following equalities hold

$$\frac{\partial l_q(x)}{\partial x} A_q(x) = \mathcal{A}_q \phi_q^T(x),$$

$$\frac{\partial l_q(x)}{\partial x} [A_q^T(x)]^T = \mathcal{B}_q \phi_q^T(x), \ j = 1, \ldots, n_q^B,$n_q^B,$

$$\bar{Y}_q(x)[A_q^T(x)]^T = \mathcal{C}_q \phi_q^T(x), \quad \bar{Y}_q(x)[\sigma_q(x)]^{1/2} = \mathcal{D}_q \phi_q^T(x),$$

$$\bar{Y}_q(x) = \mathcal{E}_q \phi_q^T(x)$$

for $x \in D_q, \ q \in Q$. Then it is easy to see that (19) holds if and only if

$$\begin{cases} \mathcal{A}_q \Lambda_q^T(x) + \mathcal{C}_q \Lambda_q^T(x) \bar{Y}_q + \mathcal{E}_q \Lambda_q^T(x) \bar{Y}_q \bar{Y}_q^{1/2} = 0, \\ \frac{1}{2} \mathcal{B}_q \Lambda_q^T(x) + \mathcal{E}_q \Lambda_q^T(x) \bar{Y}_q \bar{Y}_q^{1/2} = 0, \quad j = 1, \ldots, n_q^B, \\ -\mathcal{D}_q \Lambda_q^T(x) + \mathcal{E}_q \Lambda_q^T(x) \bar{Y}_q^{1/2} = 0, \quad j = 1, \ldots, n_q^B, \\ \mathcal{E}_q \Lambda_q^T(x) \bar{Y}_q \bar{Y}_q^{1/2} = 0, \quad j = 1, \ldots, n_q^B, i \neq j. \end{cases}$$

Lastly, consider the condition (16). For $e \in E$, let $l_{e_1}(\xi)$ and $\Lambda_{e_1}(\xi)$ be a pair of a vector-valued function and a map that provide a sum of squares by $\mu_{e_1}(x) = \langle l_{e_1}(x), \Lambda_{e_1}(x) \rangle$ with a positive semidefinite matrix $\bar{Y}_{e_1}$. Define matrices $\mathcal{T}_{e_1}, \mathcal{G}_{e_1}, \mathcal{H}_{e_1}$ and a vector-valued function $\phi_{e_1}(\xi)$ with linearly independent elements satisfying $l_{e_1} \circ \eta^{(d)}(\xi) = \mathcal{T}_{e_1} \phi_{e_1}(\xi), \ l_{e_1} \circ \eta^{(d)} \circ \eta^{(d)}(\xi) = \mathcal{G}_{e_1} \phi_{e_1}(\xi)$ and $\bar{Y}_{e_1}(\xi) = \mathcal{H}_{e_1} \phi_{e_1}(\xi)$ for $\xi \in G^{d}, \ j = 1, \ldots, N_{e_1}, \ e = (q, r) \in E$. Then $V_q(x) - V_r(R_q^N(x)) = \langle l_{e_1}(x), \Lambda_{e_1}(x) \rangle$ holds for all $x \in G^{d}$ if and only if

$$\mathcal{T}_{e_1} \Lambda_{e_1}^T(x) - \mathcal{G}_{e_1} \Lambda_{e_1}^T(x) = \mathcal{H}_{e_1} \Lambda_{e_1}^T(x) \bar{Y}_{e_1}(\xi)$$

(21)

for $j = 1, \ldots, N_{e_1}, \ e = (q, r) \in E$.

Remark 4: If $R_q^N(x)$ is an identity map, i.e., the continuous
variable does not change in discrete evolutions of $e$, then it is often natural to set $\mu_e(\xi) = 0$, by which the Lyapunov function is continuous at those discrete evolutions.

We summarize the result:

**Theorem 1:** Suppose that (20) and (21) hold for positive semidefinite matrices $Y_q$, $Y_q$, $\bar{Y}_q$, $q \in \mathcal{Q}$ and $\bar{Y}_q$, $j = 1, \ldots, N_{G_q}$, $e = (r, q) \in E$. Then $V_q(x) = \langle \rho_q(x), \Lambda_q^r(Y_q) \rangle \geq 0$ and $\sigma_q(x) = \langle \bar{\rho}_q(x), \bar{\Lambda}_q^r(\bar{Y}_q) \rangle \geq 0$ satisfy (15) and (16).

Thus the stability analysis problem is reduced to a standard semidefinite programming problem, which enjoys various fast algorithms.

### 4. Numerical Example

Let us consider the following hybrid automaton:

**Q** = $\mathcal{Q}_0 = \{+,-\}$, $\mathbf{X}_r = \mathbf{X}_- = \mathbb{R}^2$,

**Init** = \{[(+,$x$) : $x \in \mathbb{R}^2, x_1 > 0$] ,

$f^B_{l}$($x$) = $\begin{bmatrix} -x_1 - x_2 - x_1^2 - x_2^2 - x_1 x_2^2 - x_2^3 \\ 2x_1 - x_2 + x_1^3 - x_2^3 + x_1 x_2^2 - x_2^2 \end{bmatrix}$,

$f^B_{j}$($x$) = $\begin{bmatrix} -0.5x_1 - 4x_2 + 0.5x_1^2 + x_1 x_2^2 - x_2^3 \\ -x_1^3 x_2 - x_1 x_2^2 - 4x_2^3 + x_1 x_2^2 \end{bmatrix}$,

$f^B_{k}$($x$) = $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

$f^C_{l}$($x$) = $f^C_{j}$($x$) = $x_1$, $D_+$ = \{ $x : x_1 \geq 0$\}, $D_-$ = \{ $x : x_1 \leq 0$\},

$E = \{e_1 = (-, +), e_2 = (+, -)\}$,

$\mathcal{G}_1 = \mathcal{G}_2 = \{ x : x_1 = 0 \}$,

$\mathcal{R}^l_{q_1}(x) = \mathcal{R}^l_{q_2}(x) = x$.

In this section the $i$-th element of $x$ is denoted by $x_i$. We seek a Lyapunov function and find an upper bound of $k_+$ and $k_-$ for which $\text{Equi}$ is stable. We choose basis for $V_q(x)$, $\sigma_q(x)$ and $W_q(x)$ as

$\Omega_q(x) = \begin{bmatrix} x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 \end{bmatrix}$,

$\Omega_{\bar{q}}(x) = \begin{bmatrix} x_1 & x_2 & 1 \end{bmatrix}$,

$\hat{\Omega}_q(x) = \begin{bmatrix} x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 & x_1^3 & x_1 x_2^2 & x_2^3 & x_1 x_2 x_2 & x_1 x_2^2 & x_2^3 & x_1 x_2 x_2 & x_2 & 1 \end{bmatrix}$

for $q \in \mathcal{Q}$, respectively, while we set $\mu_e(\xi) = 0$. A Lyapunov function is obtained as

$V_q(x) = x_1^2 - 0.40x_1 x_2 + 0.04x_1^2 x_2^2 + 0.22x_1^2$

$-0.12x_1 x_2 + 0.23x_1^2 x_2 + 0.40x_1 x_2^2 + 0.12x_2^4$,

$V_\bar{q}(x) = 0.30x_1^2 - 0.40x_1 x_2 + 0.25x_2^2 + 0.074x_1^3$

$-0.31x_1 x_2 + 0.45x_1 x_2^2 - 0.22x_1 x_2^2 + 0.12x_2^4$,

with

\[ \sigma_+(x) = 0.40x_1^2 + 0.014x_1 x_2 + 0.36x_2^2 + 0.39 \]

\[ \sigma_-(x) = 0.061x_1^2 - 0.015x_1 x_2 + 0.32x_2^2 + 0.105 \]

and an upper bound of $k_q$ is derived as $k^* = 1.9975$. We plot $x_q(t)$ of the execution satisfying (13) in Fig.1 for $t_0 = 0$, $q_0 = +$, $x_0(t_0) = [0.7, 0.7]^T$ with setting

![Fig. 1 An execution and level surfaces of the derived Lyapunov function.](image1)

![Fig. 2 Uncertainty profile.](image2)

![Fig. 3 Response of the derived Lyapunov function.](image3)
\[ \Delta_x(t, x) = k^* e^{-(t/\beta)} \cos(t/2) \] and \[ \Delta_y(t, x) = k^* \sin(t/2), \]
shown as in Fig. 2. Level surfaces of the derived Lyapunov function \( V_q(x) \) are shown in Fig. 1, while \( V_{qk}(x_k(t)) \) is plotted in Fig. 3.

5. Conclusion

We have shown a method of computing Lyapunov functions for stability analysis of hybrid automata with set-valued vector fields. We generalized the formulation of scalar-valued sums of squares in [6] to matrix-valued ones and applied them to derive an LMI/LME problem whose solution yields a Lyapunov function.

References


