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A Non-Boolean Lattice Derived by Double Indiscernibility.

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Non-Boolean Lattice Derived by Double Indiscernibility

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The central notion of a rough set is indiscernibility based on equivalence relation. Since equivalence relation shows strong bondage in an equivalence class, it forms a Galois connection and the difference between upper and lower approximations is lost. We here introduce two different equivalence relations, the one for upper approximation, and the other for lower approximation, and construct composite approximation operator consisting of different equivalence relations. We show that a collection of fixed points with respect to the operator is a lattice, and that there exists a representation theorem for that construction.

1. Introduction

This paper is written to make a difference between topological space and rough set theory [1, 2] clear in a term of lattice theory [3, 4]. Rough set provides a method for data analysis, based on the notion of indiscernibility which is defined by equivalence relation [5, 6]. Since equivalence classes can be analogously used as open sets, similar notions used in topological space can be defined. Upper and lower approximations in a rough set theory correspond to closure and internal set, respectively [7, 8].

On one hand, closure and internal set are defined under the constraint of a topological space (i.e., closed with respect to finite intersection and to any union). On the other hand, upper and lower approximation can be defined independent of such a kind of constraint. The essential difference between operations in a topological space and approximations in a rough set is the relationship between an element and a set (open set or equivalence class) containing the element. Any elements in an equivalence class have the same equivalence class, different from the case of topological space. This strong property of equivalence class leads to a fixed point with respect to approximation that is not found in a closure operator in a topological space.

Logical structure of rough set has been studied in terms of modal logic [7, 8, 9] and lattice theory [10, 11, 12, 13, 14]. A topological space equipped with closure leads to modal logic. A rough set equipped with approximation similarly leads to modal logic, while approximation operator is re-defined by modal style binary relation different from original approximation in a rough set [15, 16]. A lattice is an ordered set that is closed with respect to join and meet, and turns to be useful in computer science [4, 12]. Recently a lattice is constructed based on approximation operator of a rough set, while equivalence relation is generalized with binary relation [11, 12, 14, 16] (Note that [12] refers to a lattice derived by a specific operator defined by equivalence relation). Due to generalization, indiscernibility is lost, and one cannot examine how equivalence relation plays a role in a lattice structure.

We study here approximations and rough sets defined by equivalence relation. Especially we introduce plural different equivalence relations and define a pseudo-closure operator based on upper and lower approximations of different equivalence relations. A lattice is defined by a collection of fixed point of pseudo-closure operator. We show that if the one equivalence relation is included by the other then a derived lattice is a set lattice that is Boolean, and otherwise a derived lattice is not restricted to a Boolean lattice. We finally show that any lattice can be expressed as a collection of fixed points of pseudo-closure operator. Existence of plural different indiscernibility plays a role in diversity of lattices.

2. Lattice derived by a single indiscernibility

First we review the necessary tools defined in a rough set theory [1, 2, 5].

Definition 1. Rough Set

Given a universal set U, let $R \subseteq U \times U$ be an equivalence relation on U. For $X \subseteq U$, we define the R-upper and R-lower approximation of X, denoted by $R^*(X)$, $R_*(X)$, respectively, as follow,

$$R^*(X) = \{x \in U \mid [x]_R \cap X \neq \varnothing\}$$
$$R_*(X) = \{x \in U \mid [x]_R \subseteq X\},$$

where $[x]_R$ is an equivalence class of R such that $[x]_R = \{y \in U \mid xRy\}$.

The difference between upper and lower approximations produces a boundary of an object X, dependent on a given equivalence relation. The lower and upper approximations mimic internal set and closure operator in a topological space, respectively. If U is a topological space, $X \subseteq U$ and an open set containing an element x is expressed as O_x, closure operator is expressed as $C(X) = \{x\in U \mid O_x\cap X\neq\emptyset\}$, and internal set as $X^{\mathrm{int}} = \{x\in U \mid O_x\subseteq X\}$. Although in an open set $y\in U_x$ does not imply $U_x = U_y$, $y\in[x]_R$ implies $[x]_R=[y]_R$ in an equivalence class. This difference is the central notion of indiscernibility.

Basic properties of approximations are listed by the following. They are used in defining a pseudo-closure operator and a lattice derived by approximations.

Theorem 2. Basic Properties of Rough Set ([2])

If $R\subseteq U\times U$ is an equivalence relation on U, for a subset $X\subseteq U$, the following statements hold.

(i) $R_*(X)\subseteq X\subseteq R^*(X)$

(ii) $R^*(\emptyset) = R_*(\emptyset) = \emptyset$, $R^*(U) = R_*(U) = U$

(iii) $R_*(X\cap Y) = R_*(X)\cap R_*(Y)$, $R^*(X\cup Y) = R^*(X)\cup R^*(Y)$

(iv) $X\subseteq Y \Rightarrow R_*(X)\subseteq R_*(Y)$, $R^*(X)\subseteq R^*(Y)$

(v) $R_*(X\cup Y)\supseteq R_*(X)\cup R_*(Y)$, $R^*(X\cap Y)\subseteq R^*(X)\cap R^*(Y)$

(vi) $R_*(U-X) = U- R^*(X)$, $R^*(U-X) = U- R_*(X)$

(vii) $R_*(R_*(X)) = R^*(R_*(X)) = R_*(X)$

(viii) $R^*(R^*(X)) = R_*(R^*(X)) = R^*(X)$

These statements mimic the properties of closure and internal set in a topological space. The statement like $R^*(R_*(X)) = R_*(X)$ and $R_*(R^*(X)) = R^*(X)$ is not found in a topological space. They hold due to the property that $y\in[x]_R$ implies $[x]_R=[y]_R$ In fact, it is easy to see that $R_*(X)\subseteq R^*(R_*(X))$. Conversely, in supposing $x\in R^*(R_*(X))$, $[x]_R\cap R_*(X)\neq\emptyset \Leftrightarrow {}^\exists y\in[x]_R$, $y\in R_*(X) \Leftrightarrow {}^\exists y\in[x]_R$, $[y]_R\subseteq X$. Since $y\in[x]_R$ implies that $[x]_R=[y]_R$, $[x]_R\subseteq X$ and $x\in R_*(X)$. Finally we obtain $R_*(X)=R^*(R_*(X))$. This kind of operation plays a central role in generating a non-Boolean lattice, given plural equivalence relations.

It is easy to see that upper and lower closure forms a Galois connection. It is shown also in [12], and we show some properties of Galois connection.

Theorem 3. Galois Connection

Given a universal set U, and an equivalence relation $R\subseteq U\times U$, for a subset X, $Y\subseteq U$, the following, Galois connection holds:

$$R^*(X)\subseteq Y \quad\Leftrightarrow\quad X\subseteq R_*(Y).$$

Proof

It is proved in [10].

Theorem 4. Properties of Galois Connection

Given a universal set U, and an equivalence relation $R \subseteq U \times U$**,** for a subset $X \subseteq U$,

$$R^*(R_*(X)) \subseteq X, \qquad X \subseteq R_*(R^*(X))$$

holds.

Proof.

It is proved in [17] that $R_*R^*: \wp(U) \to \wp(U)$ is a lattice-theoretical closure operator and $R^*R_*: \wp(U) \to \wp(U)$ is a lattice-theoretical interior operator.

Theorem 5. Duality of Fixed Points

Given a universal set U, and an equivalence class $R \subseteq U \times U$**,** for a subset $X \subseteq U$,

$$R^*(X)=X \quad \Leftrightarrow \quad R_*(X)=X.$$

Proof.

It is proved in [10].

Due to the duality of a fixed point a collection of a fixed point of $R_*(X)=X$ forms a set lattice that is a Boolean lattice.

Theorem 6. Lattice Derived from a Single Equivalence Class

Given a universal set U, and an equivalence class $R \subseteq U \times U$**,** a partially ordered set ordered by inclusion such as $<P; \subseteq>$ with $P=\{X \subseteq U \mid R_*(X)=X\}$, is a set lattice. Similarly, $<Q; \subseteq>$ with $Q=\{X \subseteq U \mid R^*(X)=X\}$ is also a set lattice.

Proof

It is proved in [11] that if (f, g) is a Galois connection on a complete Boolean lattice such that f is extensive and self-conjugate, then the set of fixed points of f forms a complete lattice. This is now the case.

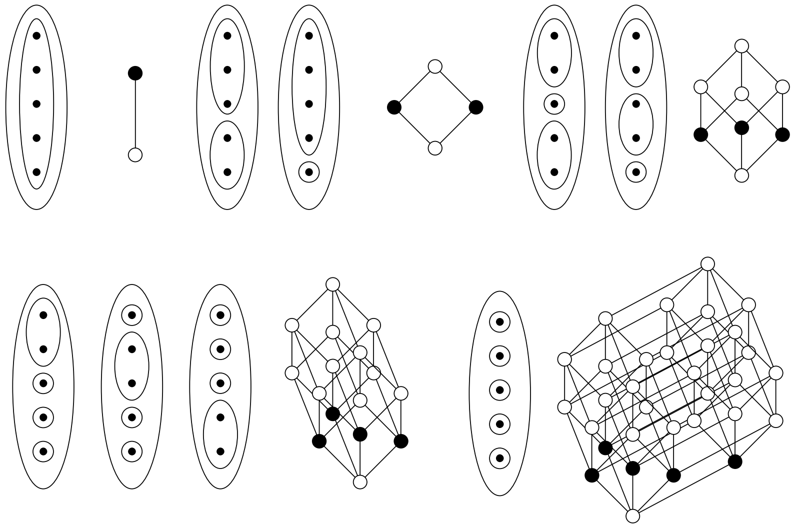

Fig. 1. The diagram of nested loop represents an equivalence relation in a set consisting of five elements. Inner loops represent equivalence classes. Right handed diagram of nested loop is a Hasse diagram of a lattice defined by $\langle P=\{X\subseteq U \mid R_*(X)=X\}; \subseteq\rangle$ from left handed equivalence relation. The number of equivalence classes, n, forms a $\mathbf{2}^n$-Boolean lattice that is a set lattice. An equivalence class corresponds to an atom represented by a black circle in each Hasse diagram.

As shown in Fig. 1, a lattice defined in theorem 6 is a $\mathbf{2}^n$-Boolean lattice, where n is the number of equivalence classes. Each equivalence class corresponds to an atom of Boolean lattice, and any union of equivalence classes exist in a lattice. In the next section, we introduce pseudo-closure operator and examine the gap between upper and lower approximations.

3. Lattice derived by ordered indiscernibility

As mentioned above, the difference between upper and lower approximations produces a boundary of an object, X, dependent on an equivalence relation. Indiscernibility takes a central part in forming a “thick” boundary. We are interested in how such a thick boundary contributes a lattice structure. To estimate the role of thick boundary, we introduce the composition of upper and lower approximation. In this section we introduce two equivalence relations, where the one relation is included by the other. It contains the case of single equivalence relation such as included by itself. Ordered indiscernibility is studied also under the name of dependency in [18].

Definition 7. Order of Equivalence Relation

Given a universal set U, and equivalence relations R, $S\subseteq U\times U$, we define an order of equivalence relation $R\leq S$, if for any $x, y\in U$, $xRy\Rightarrow xSy$.

Proposition 8. Properties of Equivalence Relation with Order

Given a universal set U, equivalence relations R, $S\subseteq U\times U$, suppose $R\leq S$. Then the following statements hold.

(i) $S_*(X)\subseteq R_*(X)$, $R^*(X)\subseteq S^*(X)$

(ii) $S^*(X)\subseteq R_*(S^*(X))$, $R^*(S_*(X))\subseteq S_*(X)$

Proof

(i) We would prove $x\in S_*(X)\Rightarrow x\in R_*(X)$. The statement is equivalent to: $[x]_S\subseteq X\Rightarrow [x]_R\subseteq X$. Then under the assumption $[x]_S\subseteq X$, we would prove $[x]_R\subseteq X$. Supposing $y\in[x]_R$, xRy. From $R\leq S$, we obtain xSy and then $y\in[x]_S$. Because of the assumption $[x]_S\subseteq X$ and $y\in[x]_S$, $y\in X$. Finally we obtain $[x]_R\subseteq X$. Similarly, we would prove $R^*(X)\subseteq S^*(X)$, that is $x\in R^*(X)\Rightarrow x\in S^*(X)$. From the assumption such that $[x]_R\cap X\neq\emptyset$, $^\exists y\in[x]_R$, $y\in X$. Then we obtain xRy and from $R\leq S$, xSy. It means $y\in[x]_S$. Thus, there exists $y\in X$ such that $y\in[x]_S$. Finally we obtain $[x]_S\cap X\neq\emptyset$.

(ii) From (i) and Theorem 2-(vii), $S^*(X) = S_*(S^*(X)) \subseteq R_*(S^*(X))$. Similarly, $R^*(S_*(X))\subseteq S^*(S_*(X)) = S_*(X)$

Theorem 9. Lattice Ordered by Equivalence Relations

Given a universal set U, an order of equivalence relations R, $S\subseteq U\times U$ is defined by $R\leq S$. Then $<P; \subseteq>$ with $P=\{X\subseteq U|\ R_*(S^*(X))=X\}$ is a set lattice. Similarly, $<Q; \subseteq>$ with $Q=\{X\subseteq U|\ R^*(S_*(X))=X\}$ is also a set lattice. The notation $<P; R>$ implies a partially ordered set P ordered by relation R.

Proof. From theorem 2-(i), $R_*(S^*(X))\subseteq S^*(X)$, and from proposition 8-(ii), $S^*(X)\subseteq R_*(S^*(X))$. Thus $S^*(X)=R_*(S^*(X))$. Since $R_*(S^*(X))=X$ in P, we obtain $S^*(X)=X$ in P. It means a lattice $<P; \subseteq>$ with $P=\{X\subseteq U|\ R_*(S^*(X))=X\}$ is equivalent to $<\{X\subseteq U|\ S^*(X)=X\}; \subseteq>$ that is a set lattice from Theorem 6. Similarly, it is easy to see that $<Q; \subseteq>$ with $Q=\{X\subseteq U|\ R^*(S_*(X))=X\}$ is equivalent to $<\{X\subseteq U|$

$S_*(X)=X\}$; $\subseteq$> that is a set lattice.

From theorem 9, if two equivalence relations are ordered, composition of two approximations is reduced to one approximation. Thus the effect of “thick” boundary of an object is lost, and the derived lattice consists of all unions of equivalence classes.

Corollary 10. Lattice driven by Single Equivalence Relations

Given a universal set U, $R\subseteq U\times U$ is defined as an equivalence relation. Then <P; $\subseteq$> with $P=\{X\subseteq U| R_*(R^*(X))=X\}$ is a set lattice. Similarly, <Q; $\subseteq$> with $Q=\{X\subseteq U| R^*(R_*(X))=X\}$ is also a set lattice.

Proof. It is easy to prove it from theorem 9, since $R\leq R$.

Corollary 10 shows that composition of lower and upper approximations is reduced to a single approximation. Even if objects are recognized dependent on the approximation based on an equivalence relation, structure of a lattice is invariant. That is Boolean lattice. How is diversity of lattice structure arisen? In the next section we introduce two equivalence relations that are not ordered, and show that composition of upper and lower approximations cannot be reduced to one.

4. Lattice driven by plural indiscernibility

In this section, we introduce two non-ordered equivalence relations and the operator that is a composition of upper and lower approximations, where the upper approximation is based on the one relation and the lower one is based on the other relation. First we examine this operator in a term of closure operator. In fact, since this operator doe not satisfy all of the conditions of closure operator, we call this pseudo-closure operator.

Theorem 11. Pseudo-Closure Derived from Plural Equivalence Relations

Given a universal set U, R and $S\subseteq U\times U$ are defined as different equivalence relations. The operations T and S are defined by $T=S_*R^*$, $K= R^*S_*$. Then, for X, $Y\subseteq U$,

(i) $X \subseteq Y \;\Rightarrow\; T(X) \subseteq T(Y)$, $K(X) \subseteq K(Y)$

(ii) $T(T(X)) = T(X)$, $K(K(X)) = K(X)$,

Proof.

(i) Supposing $x\in T(X)$, from Theorem 2-(vii), $x\in S_*(R^*(X)) \Leftrightarrow [x]_S\subseteq R^*(X) \Leftrightarrow {}^\forall y\in[x]_S$, $[y]_R\cap X\neq\emptyset \Leftrightarrow {}^\forall y\in[x]_S$, ${}^\exists z\in[y]_R$, $z\in X$. From the assumption $z\in X\subseteq Y$, ${}^\forall y\in[x]_S$, ${}^\exists z\in[y]_R$, $z\in Y$ $\Leftrightarrow {}^\forall y\in[x]_S$, $[y]_R\cap Y\neq\emptyset \Leftrightarrow [x]_S\subseteq R^*(Y) \Leftrightarrow x\in S_*(R^*(Y))=T(Y)$.Similarly, supposing $x\in K(X)$, $x\in R^*(S_*(X)) \Leftrightarrow [x]_R\cap S_*(X)\neq\emptyset \Leftrightarrow {}^\exists y\in[x]_R$, $[y]_S\subseteq X$. From the assumption, ${}^\exists y\in[x]_R$, $[y]_S\subseteq Y\Leftrightarrow$ $y\in[x]_R$, $y\in S_*(Y) \Leftrightarrow [x]_R\cap S_*(Y)\neq\emptyset \Leftrightarrow x\in R^*(S_*(Y))=K(Y)$.

(ii) Supposing $x\in T(T(X))$, then

$$
\begin{aligned}
& x\in S_*(R^*(S_*(R^*(X)))) \\
\Leftrightarrow\ & [x]_S\subseteq R^*(S_*(R^*(X))) \Leftrightarrow {}^\forall y\in[x]_S,\ [y]_R\cap S_*(R^*(X))\neq\emptyset \\
\Leftrightarrow\ & {}^\forall y\in[x]_S,\ {}^\exists z\in[y]_R,\ [z]_S\subseteq R^*(X) \Leftrightarrow {}^\forall y\in[x]_S,\ {}^\exists z\in[y]_R,\ {}^\forall w\in[z]_S,\ [w]_R\cap X\neq\emptyset.
\end{aligned}
$$

From $z\in[y]_R$, $[y]_R= [z]_R$. Since it is trivially true that $z\in[z]_S$, z can be applied to the statement, $w\in[z]_S$, $[w]_R\cap X\neq\emptyset$. Thus we obtain $[z]_R\cap X\neq\emptyset$, and then $[y]_R\cap X\neq\emptyset$. Finally,

$$
\begin{aligned}
& x\in S_*(R^*(S_*(R^*(X)))) \\
\Leftrightarrow\ & {}^\forall y\in[x]_S,\ {}^\exists z\in[y]_R,\ {}^\forall w\in[z]_S,\ [w]_R\cap X\neq\emptyset \Rightarrow {}^\forall y\in[x]_S,\ [y]_R\cap X\neq\emptyset \\
\Leftrightarrow\ & [x]_S\subseteq R^*(X) \Leftrightarrow x\in S_*(R^*(X)) \Leftrightarrow x\in T(X)
\end{aligned}
$$

Conversely, supposing $x\in T(X)$, $x\in S_*(R^*(X)) \Leftrightarrow [x]_S\subseteq R^*(X) \Leftrightarrow {}^\forall y\in[x]_S$, $[y]_R\cap X\neq\emptyset$. Since $y\in[y]_R$, there exists an element both in $[x]_S$ and $[y]_R$. It implies ${}^\exists z\in[y]_R, z\in[x]_S$, and then $[x]_S=[z]_S$. Thus, in the statement ${}^\forall y\in[x]_S$, $[y]_R\cap X\neq\emptyset$, we can replace $[x]_S$ by $[z]_S$, and y by w. Therefore, the statement ${}^\forall w\in[z]_S$, $[w]_R\cap X\neq\emptyset$ holds under the condition ${}^\exists z\in[y]_R$. It follows:

$$
\begin{aligned}
& x\in S_*(R^*(X)) \\
\Leftrightarrow\ & {}^\forall y\in[x]_S,\ [y]_R\cap X\neq\emptyset \\
\Rightarrow\ & {}^\forall y\in[x]_S,\ {}^\exists z\in[y]_R,\ {}^\forall w\in[z]_S,\ [w]_R\cap X\neq\emptyset \\
\Leftrightarrow\ & {}^\forall y\in[x]_S,\ {}^\exists z\in[y]_R,\ {}^\forall w\in[z]_S,\ w\in R^*(X) \\
\Leftrightarrow\ & x\in S_*(R^*(S_*(R^*(X)))) \quad\Leftrightarrow\quad x\in T(T(X)).
\end{aligned}
$$

We introduce a lattice as a collection of fixed points with respect to pseudo-closure operators. Since join and meet of a lattice is not defined by union and intersection, information with respect to combinations of equivalence classes are lost in a derived lattice. Thus we can see not only Boolean but any other various lattices.

Theorem 12. Lattice driven by Pseudo-Closure

Given a universal set U, R and $S \subseteq U \times U$ are defined as different equivalence relations, and $T = S_* R^*$ and $K = R^* S_*$. A partially ordered set $<\boldsymbol{L}_T; \subseteq>$ with $\boldsymbol{L}_T = \{X \subseteq U \mid T(X) = X\}$ is a lattice. Similarly, $<\boldsymbol{L}_K; \subseteq>$ with $\boldsymbol{L}_K = \{X \subseteq U \mid K(X) = X\}$ is also a lattice. Meet and join of a lattice is defined by: for $X, Y \in \boldsymbol{L}_T$

$$X \wedge Y = T(X \cap Y), \qquad X \vee Y = T(X \cup Y).$$

Similarly, for $X, Y \in \boldsymbol{L}_K$

$$X \wedge Y = K(X \cap Y), \qquad X \vee Y = K(X \cup Y).$$

Indeed, $<\boldsymbol{L}_T; \subseteq>$ and $<\boldsymbol{L}_K; \subseteq>$ are complete lattices.

Proof

We would prove that $<\boldsymbol{L}_T; \subseteq>$ with $\boldsymbol{L}_T = \{X \subseteq U \mid T(X) = X\}$ is a lattice. The statement on the operation K can be proven by a similar manner.

(i) First we would check that meet and join are well-defined. Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, applying $T = S_* R^*$ to these inequalities leads to that $T(X \cap Y) \subseteq T(X)$ and $T(X \cap Y) \subseteq T(Y)$, because of theorem 2-(iv). Thus we obtain $X \wedge Y = T(X \cap Y) \subseteq T(X) = X$ and $X \wedge Y = T(X \cap Y) \subseteq T(Y) = Y$, and that implies that $X \wedge Y$ is a lower bound of $\{X, Y\}$. Supposing that $Z \in \boldsymbol{L}_T$ is a lower bound of $\{X, Y\}$, we obtain $Z \subseteq X$ and $Z \subseteq Y$, and then $Z \subseteq X \cap Y$. It leads to that $T(Z) \subseteq T(X \cap Y)$ and $Z = T(Z) \subseteq T(X \cap Y) = X \wedge Y$. It implies that $X \wedge Y$ is the greatest lower bound. Similarly, from $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$, we obtain $X = T(X) \subseteq T(X \cup Y) = X \vee Y$ and $Y = T(Y) \subseteq T(X \cup Y) = X \vee Y$, and that implies $X \vee Y$ is an upper bound of $\{X, Y\}$. Supposing that $Z \in \boldsymbol{L}_T$ is an upper bound, $X \subseteq Z$, $Y \subseteq Z$ and $X \cup Y \subseteq Z$, and then $X \vee Y = T(X \cup Y) \subseteq T(Z) = Z$. It implies that $X \vee Y$ is the least upper bound.

(ii) We would prove that a partially ordered set $<\boldsymbol{L}_T; \subseteq>$ is closed with respect to join and meet. In order to show $X \wedge Y \in \boldsymbol{L}_T$, we have to show $T(X \cap Y) \in \boldsymbol{L}_T$. In fact, $T(X \wedge Y) = T(T(X \cap Y)) = T(X \cap Y) = X \wedge Y$. Thus, $X \wedge Y \in \boldsymbol{L}_T$. Similarly, $T(X \vee Y) = T(T(X \cup Y)) = T(X \cup Y) = X \vee Y$, and then $X \vee Y \in \boldsymbol{L}_T$.

(iii) For any subset $M \subseteq \boldsymbol{L}_T$, $T(\wedge M) = TT(\cap M)$. For any $X_i \in M$, $TT(X_i) = T(X_i)$, and then $TT(\cap M) = T(\cap M) = \wedge M$. Thus we obtain $\wedge M \in \boldsymbol{L}_T$. Similarly we obtain $\vee M \in \boldsymbol{L}_T$. It can be verified straightforwardly that $<\boldsymbol{L}_K; \subseteq>$ is also a complete lattice.

The next proposition shows an element of a lattice in theorem 12 is expressed as a union of equivalence classes based on a single equivalence relation.

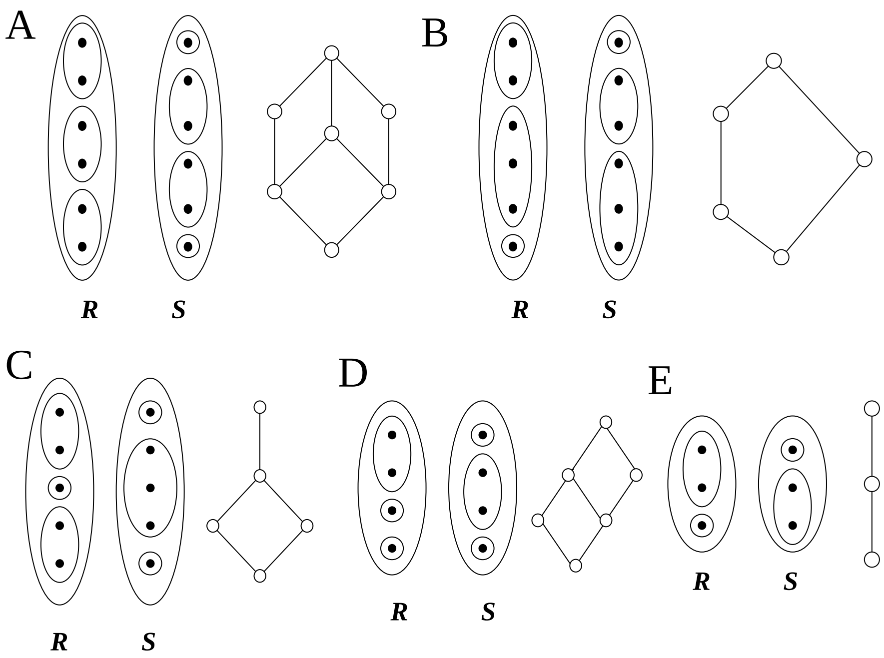

Fig. 2. The diagram of nested loop represents an equivalence relation R or S in a universal set. A lattice derived from a pair of equivalence relations, R and S is expressed by Hasse diagram. Each lattice is defined by $<\boldsymbol{L}_T; \subseteq>$ with $\boldsymbol{L}_T = \{X \subseteq U | T(X)=X\}$. Each elements of a lattice is expressed as a union of equivalence classes of S.

Lemma 13. Elements of a lattice

Given a universal set U, R and $S \subseteq U \times U$ are defined as different equivalence relations. The operations T and S are defined by $T=S_*R^*$, $K= R^*S_*$. Then, for $X \subseteq U$,

(i) $T(X) = X \quad \Rightarrow \quad S_*(X) = X$,

(ii) $K(X) = X \quad \Rightarrow \quad R^*(X) = X$.

Proof

(i) Supposing $S_*R^*(X) = X$, we obtain $S_*S_*R^*(X) = S_*(X)$. From theorem 2-(vii), left handed form $S_*S_*R^*(X) = S_*R^*(X)$. Then, $S_*R^*(X) = S_*(X)$, and finally $S_*(X) = X$.

(ii) As well as (i), it is easy to obtain $R^*(X) = X$ by using theorem 2-(viii).

Theorem 14. Galois connection in a derived lattice

Given a universal set U, R and $S \subseteq U \times U$ are defined as different equivalence relations. Then, for $X, Y \in \boldsymbol{L}_T = \{X \subseteq U \mid T(X) = X\}$, the following Galois connection holds:

$$R^*(X) \subseteq Y \quad \Leftrightarrow \quad X \subseteq S_*(Y).$$

Proof

In supposing $X \subseteq S_*(Y)$, we obtain $S_*(R^*(X)) \subseteq S_*(Y)$. From lemma 13, $S_*(R^*(X)) = R^*(X)$, and $S_*(Y) = Y$. Then we obtain $R^*(X) \subseteq Y$. Conversely, in supposing $R^*(X) \subseteq Y$, we obtain $S_*(R^*(X)) \subseteq S_*(Y)$ from theorem 2-(iv). Then, $X \subseteq S_*(Y)$.

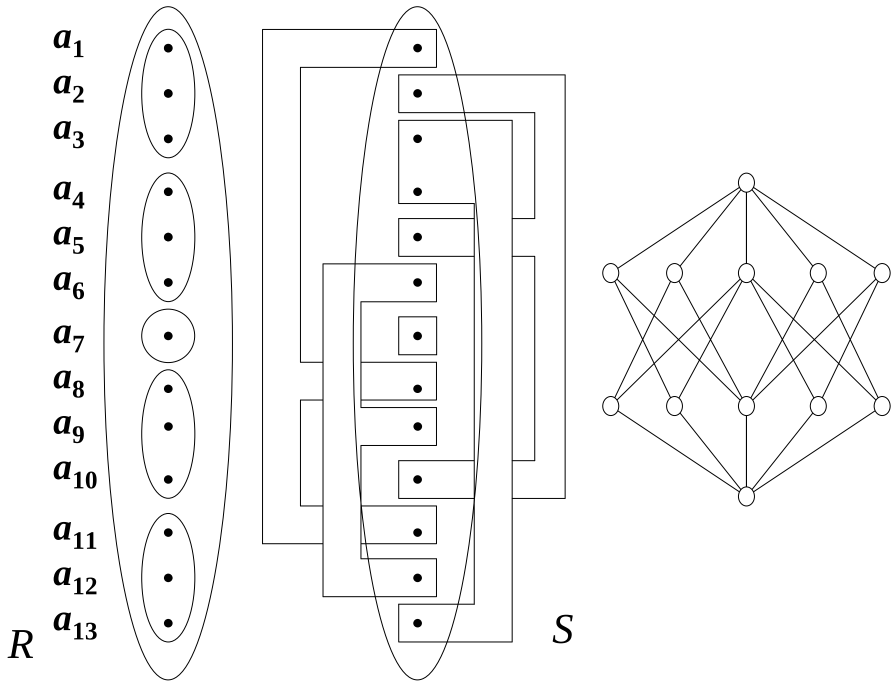

Fig. 3. Hasse diagram of an orthocomplemented lattice (right) defined by a collection of $T(X) = X$. A universal set is $\{a_1, a_2, \ldots, a_{13}\}$. Equivalence classes of R are represented by loops, and those of S are represented by polygons.

Fig. 2 shows some examples of a lattice defined in theorem 12. As mentioned in lemma 13, if a lattice is defined as a collection of a fixed point with respect to the operator T, elements of a lattice is expressed as a union of equivalence classes of S. We here write down all elements of lattices in Fig. 2A, where all elements of a universal set are denoted by a, b, c, d, e, f in a descending direction. Equivalence classes of R are $\{a, b\}$, $\{c, d\}$ and $\{e, f\}$, and equivalence classes of S are $\{a\}$, $\{b, c\}$, $\{d, e\}$ and $\{f\}$. If one collects all fixed points with respect to T, that is $\{\{a\}, \{f\}, \{a, b, c\}, \{a, f\}, \{d, e, f\}, X\}$.

More complex orthocomplemented lattice is also constructed by a collection of fixed points with respect to T, as shown in Fig. 3. Atoms of a lattice are equivalence classes of S. For a subset $\{a_1, a_8, a_{11}\}$ that is an equivalence class of S, it is easy to verify that it forms a fixed point by $S_*R^*(\{a_1, a_8, a_{11}\}) = S_*(\{a_1, a_2, a_3, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}\}) = \{a_1, a_8, a_{11}\}$.

It is easy to see that there exists a lattice isomorphism between $\boldsymbol{L}_T$ and $\boldsymbol{L}_K$ To verify that, first we show a Galois connection between $<\boldsymbol{L}_T; \subseteq>$ and $<\boldsymbol{L}_K; \subseteq>$.

Theorem 15. Galois connection between $\boldsymbol{L}_T$ and $\boldsymbol{L}_K$

Given a universal set U, R and $S \subseteq U \times U$ are defined as different equivalence relations. The pair (R^*, S_*) is a Galois connection between $<\boldsymbol{L}_T; \subseteq>$ and $<\boldsymbol{L}_K; \subseteq>$, where $T=S_*R^*$ and $K=R^*S_*$.

Proof

(i) For $X \in \boldsymbol{L}_T$ and $Y \in \boldsymbol{L}_K$, in supposing $X \subseteq S_*(Y)$, we obtain $R^*(X) \subseteq R^*(S_*(Y)) = KY = Y$. Conversely, in supposing $R^*(X) \subseteq Y$, we obtain $S_*(R^*(X)) \subseteq S_*(Y)$. Since $S_*(R^*(X)) = TX = X$, $X \subseteq S_*(Y)$. Thus, $R^*(X) \subseteq Y \Leftrightarrow X \subseteq S_*(Y)$.

Theorem 16. Lattice Isomorphism among $\boldsymbol{L}_T$, $\boldsymbol{L}_K$, $\boldsymbol{L}_N$ and $\boldsymbol{L}_M$

Given a universal set U, R and $S \subseteq U \times U$ are defined as equivalence relations, and the operators are defined by $T=S_*R^*$, $K= R^*S_*$, $M= S^*R_*$ and $N= R_*S^*$. A partially ordered set $<\boldsymbol{L}_M; \subseteq>$ with $\boldsymbol{L}_M =\{X \subseteq U \mid M(X)=X\}$ and $<\boldsymbol{L}_N; \subseteq>$ with $\boldsymbol{L}_N =\{X \subseteq U \mid N(X)=X\}$ are lattices, and all lattices are isomorphic such as;

$$<\boldsymbol{L}_T; \subseteq> \cong <\boldsymbol{L}_K; \subseteq> \cong <\boldsymbol{L}_M; \subseteq> \cong <\boldsymbol{L}_N; \subseteq>.$$

Proof

First we prove $<\boldsymbol{L}_T; \subseteq> \cong <\boldsymbol{L}_K; \subseteq>$.

(i) A map $\varphi: \boldsymbol{L}_T \to \boldsymbol{L}_K$ is defined as a lattice homomorphism, where for $X \in \boldsymbol{L}_T$, $\varphi(X) = R^*(X)$. First, we would show that $\varphi: \boldsymbol{L}_T \to \boldsymbol{L}_K$ is well-defined. For $X \in \boldsymbol{L}_T$, $S_*R^*(X)=X$. Thus, $\varphi(X) = R^*(X) = R^*(S_*R^*(X)) = K(R^*(X)) = K(\varphi(X))$. It implies that $\varphi(X) \in \boldsymbol{L}_K$. Indeed, $\varphi(X \vee Y) =$

$\varphi(T(X\cup Y)) = R^*S_*R^*(X\cup Y) = R^*S_*(R^*(X)\cup R^*(Y)) = K(\varphi(X)\cup\varphi(Y)) = \varphi(X)\vee\varphi(Y)$. Also, $\varphi(X\wedge Y) = R^*(X\wedge Y) = R^*(T(X\cap Y)) = R^*(S_*R^*(X\cap Y)) = R^*S_*(R^*(X\cap Y))$. Since $\varphi(X\wedge Y)\in \boldsymbol{P}_K$, $\varphi(X\wedge Y) = R^*(X\cap Y)$ is a fixed point with respect to R^*S_*, $R^*S_*(R^*(X\cap Y))= R^*(X\cap Y)$. Thus, $R^*(X\cap Y) = R^*(S_*R^*(X)\cap S_*R^*(Y))$, since X and $Y\in \boldsymbol{L}_T$ are fixed points with respect to S_*R^*. Therefore, from theorem 2-(iii), $R^*(S_*R^*(X)\cap S_*R^*(Y)) = R^*S_*(R^*(X)\cap R^*(Y)) = K(R^*(X)\cap R^*(Y)) = K(\varphi(X)\cap\varphi(Y)) = \varphi(X)\wedge\varphi(Y)$.

(ii) First we would show that φ: $\boldsymbol{L}_T\rightarrow \boldsymbol{L}_K$ is an injection. Supposing $\varphi(X) = \varphi(Y)$ for $X,Y\in\boldsymbol{L}_T$, $R^*(X) = R^*(Y)$. Thus, applying S_* to both sides, $S_*R^*(X) = S_*R^*(Y)$. Since $X,Y\in\boldsymbol{L}_T$, $S_*R^*(X) = X$, $S_*R^*(Y)=Y$. Thus, we obtain $X=Y$. It implies that $\varphi(X) = \varphi(Y) \Rightarrow X=Y$, and that φ is an injection. Second, we would prove that φ: $\boldsymbol{L}_T\rightarrow \boldsymbol{L}_K$ is a surjection. Since $Y\in\boldsymbol{L}_K$ is a fixed point with respect to R^*S_*, for any $Y\in\boldsymbol{L}_K$, $Y = R^*S_*(Y) = \varphi(S_*(Y))$. Also from $S_*R^*(S_*(Y)) = S_*(R^*S_*(Y)) = S_*(Y)$, $S_*(Y)$ is a fixed point with respect to S_*R^*, and then $S_*(Y)\in\boldsymbol{L}_T$. It implies that φ is a surjection.

(iii) It is easy to show that $<\boldsymbol{L}_M; \subseteq>$ and with $<\boldsymbol{L}_N; \subseteq>$ are lattices as well as theorem 9. A map $\psi:\boldsymbol{L}_T\rightarrow \boldsymbol{L}_M$ is defined by; for $X\in\boldsymbol{L}_T$, $\psi(X) = U-X$. Especially, elements in $<\boldsymbol{L}_M; \subseteq>$ are ordered by an opposite order of those in $<\boldsymbol{L}_T; \subseteq>$. Since $\psi(X) = U-X = U-S_*R^*(X) = S^*(U-R^*(X)) = S^*(R_*(U-X))\in \boldsymbol{L}_M$, the map is well-defined. If for $Y\in\boldsymbol{L}_M$, $\psi^{-1}(Y) = U-Y$, then $\psi^{-1}(Y) = U-Y = U-S^*R_*(Y) = S_*(U-R_*(Y)) = S_*(R^*(U-Y))\in \boldsymbol{L}_T$. Thus, ψ^{-1} is also well-defined. For any $X\in\boldsymbol{L}_T$, $\psi^{-1}(\psi(X)) = U-S^*(R_*(U-X)) = S_*(U-R_*(U-X)) = S_*(R^*(U-(U-X))) = S_*(R^*(X)) = X$. Thus, $\psi:\boldsymbol{L}_T\rightarrow \boldsymbol{L}_M$ is a bijection. If $X\subseteq Y$, then $U-Y\subseteq U-X$. Therefore, we obtain that $S^*(R_*(U-Y)))\subseteq S^*(R_*(U-X)))$.We would easily show that $\psi(X\wedge Y) = \psi(X)\vee\psi(Y)$, where $X\vee Y = M(X\cup Y)$ and $X\wedge Y = M(X\cap Y)$ in $<\boldsymbol{L}_M, \subseteq>$. First, $\psi(X\wedge Y) = U-X\wedge Y = U-T(X\cap Y) = U-S_*R^*(X\cap Y) = S^*(U-R^*(X\cap Y)) = S^*(R_*(U-(X\cap Y)) = M((U-X)\cup(U-Y)) = M(\psi(X)\cup\psi(Y)) = \psi(X)\vee\psi(Y)$. Next, $\psi(X\vee Y) = \psi(T(X\cup Y)) = U-S_*R^*(X\cup Y) = S^*(U-R^*(X\cup Y)) = S^*(R_*(U-(X\cup Y)) = S^*(R_*((U-X)\cap(U-Y))) = M(\psi(X)\cap\psi(Y)) = \psi(X)\wedge\psi(Y)$. Isomorphism is verified straightforwardly.

(iv) Since R^* and R_* are mutually dual, $\boldsymbol{L}_M$ is dually order isomorphic to $\boldsymbol{L}_T$. In addition, $\boldsymbol{L}_N$ is dually order isomorphic to $\boldsymbol{L}_K$ by the same argument.

Given two equivalence relations, there can be four kinds of operations consisting of upper and lower approximations, such as S_*R^*, S^*R_*, R_*S^* and R^*S_*. Theorem 16 shows that a lattice obtained as a collection of fixed points of those operators is unique up to isomorphism. Fig. 4 shows an example of four lattices as a collection of fixed points with respect to S_*R^*, S^*R_*, R_*S^* and R^*S_*. Although some of them are opposite, they are the same structure.

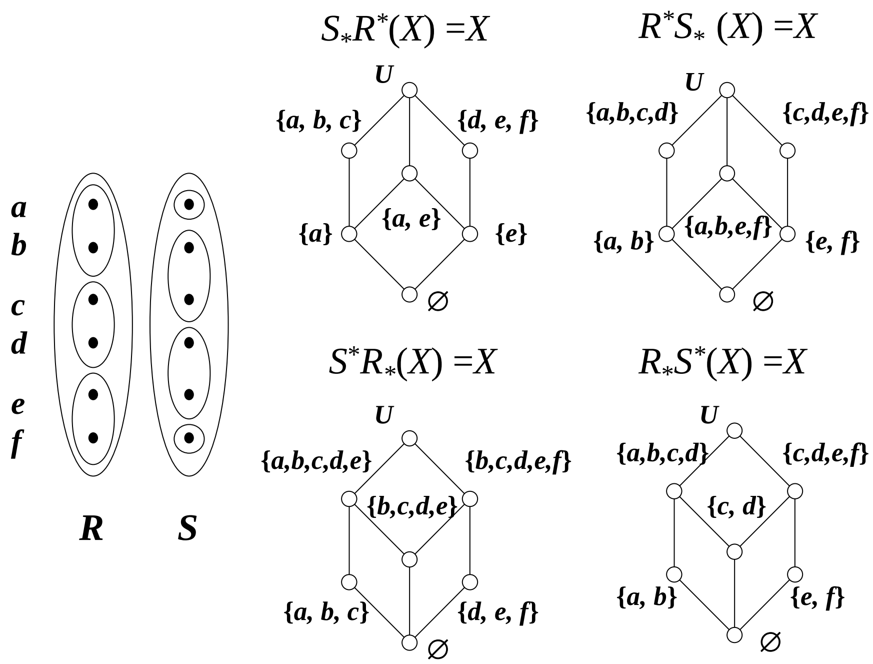

Fig.4. Given two equivalence relations R and S on a universal set U, a collection of fixed points with respect to four kinds of operators forms a unique lattice up to isomorphism.

5. Representation theorem for complete lattices in terms of double indiscernibility

In the above section we observe that two kinds of equivalence relations form a complete lattice. Conversely, we can verify any lattice can be represented by a collection of fixed points with respect to operator T. First we define a universal set and equivalence relations derived from a given lattice.

Definition 17. Universal set derived from a lattice

Let $<L; \leq>$ be a lattice. A universal set $U_L \subseteq L \times L$ derived from L is defined by $U_L = \{<x, y> \in L \times L \mid <x, y> \notin \leq\}$. Two equivalence classes derived from L, denoted by R and $S \subseteq U_L \times U_L$, are defined by $<x, y>R<z, y>$ and $<x, y>S<x, w>$.

Lemma 18.

Let $<L; \leq>$ be a lattice. Given x in L, lower and upper sets of x, $X_{x,}{}^{l}$ and $X_{x,}{}^{w}$, are defined by $X_x^{\,l} = \{<y, z>\in U_L | y\leq x\}$ and $X_x^{\,u} = \{<y, z>\in U_L | x\leq z\}$, respectively. Then

(i) $\quad R^*(X_{x,}^{\,l}) = U - X_{x,}^{\,u}$,

(ii) $\quad S_*(U - X_{x,}^{\,u}) = X_x^{\,l}$.

Proof. (i) Supposing $<y, z>\in R^*(X_x^{\,l})$, $[<y, z>]_R \cap X_x^{\,l} \neq \emptyset \Leftrightarrow {}^{\exists}<w, z>\in X_x^{\,l}$, $<w, z>\in[<y, z>]_R$. Then

$$<y, z>\notin R_x^{\,*}(X_x^{\,l}) \Leftrightarrow {}^{\forall}<w, z>\in X_x^{\,l},\ <w, z>\notin[<y, z>]_R.$$

From $<w, z>\notin[<y, z>]_R$, $w\leq z$. Since ${}^{\forall}<w, z>\in X_x^{\,l}$ implies any $w\leq x$, z is larger than the maximal w that is x. Then $x\leq z$, and

$$<y, z>\notin R_x^{\,*}(X_x^{\,l}) \Leftrightarrow <y, z>\in\{<y, z>\in U_L | x\leq z\} = X_x^{\,u}.$$

Finally we obtain $R^*(X_{x,}^{\,l}) = U - X_{x,}^{\,u}$.

(ii) Supposing $<y, z>\in S_*(U - X_{x,}^{\,u})$,

$$\begin{aligned}<y, z>\in S_*(U - X_{x,}^{\,u}) &\Leftrightarrow [<y, z>]_S \subseteq U - X_{x,}^{\,u}\\ &\Leftrightarrow {}^{\forall}<y, w>\in[<y, z>]_S \Rightarrow <y, w>\notin X_{x,}^{\,u}\\ &\Leftrightarrow {}^{\forall}<y, w>\in X_{x,}^{\,u} \Rightarrow <y, w>\notin[<y, z>]_S.\end{aligned}$$

From $<y, w>\notin[<y, z>]_S$, $y\leq w$. Since it holds for any $<y, w>\in X_{x,}^{\,u}$, i.e., for $<y, w>$ with $x\leq w$, y is smaller than the minimal w that is x. Then $y\leq x$, and

$$<y, z>\in S_*(U - X_{x,}^{\,u}) \Leftrightarrow <y, z>\in\{<y, z>\in U_L | y\leq x\} = X_x^{\,l}.$$

Finally we obtain the following representation theorem.

Theorem 19. Representation Theorem

Let L be a lattice. Then the map η:$<L; \leq>\rightarrow <\boldsymbol{L}_T; \subseteq>$ defined by

$$\eta(x) = X_x^{\,l}.$$

for $x\in L$, is an isomorphism of L onto $\boldsymbol{L}_T$, where $T=S_*R^*$, and S and $R \subseteq U_L\times U_L$, are equivalence relations on U_L, with the inverse of η given by $\eta^{-1}(X) = \vee\{y\in L | <y, z>\in X\}$ for $X\in \boldsymbol{L}_T$.

Proof. The map η is well-defined, since $T(\eta(x)) = T(X_x^l) = X_x^l$ from lemma 18. Isomorphism is verified by $\eta^{-1}(\eta(x)) = \eta^{-1}(X_x^l) = \vee\{y \in L | <y, z> \in X_x^l\} = \vee\{y \in L | y \leq x\} = x$. It can be verified straightforwardly that η is a lattice homomorphism.

Therefore, we can represent any lattice as a collection of fixed point with respect to T consisting of two equivalence relations.

5. Conclusion

We here concentrate on indiscernibility based on equivalence relations, since indiscernibility is a central notion in a rough set different from topological space. Due to indiscernibility approximation operators form a Galois connection that shows a strong bondage between two perspectives. As a result, a collection of a fixed point with respect to approximation operator forms a trivial set lattice. There is no diversity in terms of lattice structure.

Diversity of lattice structure results from discrepancy between two equivalence relations. Then we concentrate on double equivalence relations, and an operator consisting of upper approximation based on the one equivalence relation and lower approximation based on the other. Galois connection such that $R^*(X) \subseteq Y \Leftrightarrow X \subseteq S_*(Y)$ does not hold, since R is different from S. If we collect fixed points with respect to S_*R^*, $R^*(X) \subseteq Y \Leftrightarrow X \subseteq S_*(Y)$ holds in that collection (i.e., in a derived lattice). Collecting fixed points results in loss of information, and then join and meet in a lattice cannot be defined by union and intersection, respectively. It can provide a wide variety of lattices. It is verified by the existence of representation theorem.

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