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<td>Acta Mathematica Hungarica, 151(1):199-216</td>
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PDF issue: 2019-10-15
A METRIC DISCREPANCY RESULT WITH GIVEN SPEED

ISTVÁN BERKES, KATUSI FUKUYAMA, AND TAKUYA NISHIMURA

Abstract. It is known that the discrepancy $D_N\{kx\}$ of the sequence \{\{kx\}\} satisfies $ND_N\{kx\} = O((\log N)(\log \log N)^{1+\varepsilon})$ a.e. for all $\varepsilon > 0$, but not for $\varepsilon = 0$. For $n_k = \theta^k$, $\theta > 1$ we have $ND_N\{n_kx\} \leq (\Sigma_\theta + \varepsilon)(2N \log \log N)^{1/2}$ a.e. for some $0 < \Sigma_\theta < \infty$ and $N \geq N_0$ if $\varepsilon > 0$, but not for $\varepsilon < 0$. In this paper we prove, extending results of Aistleitner-Larcher [6], that for any sufficiently smooth intermediate speed $\Psi(N)$ between $(\log N)(\log \log N)^{1+\varepsilon}$ and $(N \log \log N)^{1/2}$ and for any $\Sigma > 0$, there exists a sequence \{\{n_k\}\} of positive integers such that $ND_N\{n_kx\} \leq (\Sigma + \varepsilon)\Psi(N)$ eventually holds a.e. for $\varepsilon > 0$, but not for $\varepsilon < 0$. We also consider a similar problem on the growth of trigonometric sums.

1. Introduction

A sequence \{\{x_k\}\} of real numbers is said to be uniformly distributed modulo 1 if

$$\frac{1}{N} \# \{k \leq N : \langle x_k \rangle \in [a, b)\} \to b - a, \quad (N \to \infty),$$

for all $0 \leq a < b \leq 1$, where $\langle x \rangle$ denotes the fractional part $x - \lfloor x \rfloor$ of a real number $x$. The discrepancy $D_N\{x_k\}$, also denoted by $D_N(x_1, \ldots, x_N)$, is used to measure the speed of convergence:

$$D_N\{x_k\} = \sup_{0 \leq a < b \leq 1} \left| \frac{1}{N} \# \{k \leq N : \langle x_k \rangle \in [a, b)\} - (b - a) \right|.$$

For arithmetic progressions \{\{kx\}\} with $x \notin \mathbb{Q}$, Bohl [10], Sierpiński [24], and Weyl [26] independently proved that they are uniformly distributed modulo 1. A metric result of Khintchine [20] implies

$$ND_N\{kx\} = O((\log N)(\log \log N)^{1+\varepsilon}) \quad \text{a.e. for any } \varepsilon > 0 \quad (1)$$
and this fails for \( \varepsilon \leq 0 \). The discrepancy of exponentially growing sequences has also been investigated extensively. By assuming the Hadamard gap condition

\[ n_{k+1}/n_k \geq q > 1 \quad (k = 1, 2, \ldots), \]  

(2)

Philipp [23] proved, using Takahashi’s method [25], that

\[ \frac{1}{4\sqrt{2}} \leq \lim_{N \to \infty} \frac{N D_N \{n_k x\}}{\sqrt{2 N \log \log N}} \leq \frac{1}{\sqrt{2}} \left( 166 + \frac{664}{q^{1/2} - 1} \right) \text{ a.e.} \]  

(3)

For improvements of (3), see [3] for the lower bound, and [18] for the upper bound. In case of geometric progressions, an exact law of the iterated logarithm holds: for any \( \theta \not\in [\frac{1}{2}, 1] \) there exists a constant \( \Sigma_\theta \geq \frac{1}{2} \) with

\[ \lim_{N \to \infty} \frac{N D_N \{\theta^k x\}}{\sqrt{2 N \log \log N}} = \Sigma_\theta \quad \text{a.e.} \]  

If \( \theta^j \notin \mathbb{Q} \) for any \( j \in \mathbb{N} \), then \( \Sigma_\theta = \frac{1}{2} \), otherwise \( \Sigma_\theta > \frac{1}{2} \). For a \( \theta \) which is a power root of an integer, of a large rational number, or of a ratio of odd integers, the concrete value of \( \Sigma_\theta \) is evaluated. See [12, 14, 15, 16, 17]. For conditions to have an exact law of the iterated logarithm in (3), see [1, 5].

Since there is a big difference between (1) and (3), it is natural to ask if for intermediate speeds \( \Psi(N) \) between \((\log N)(\log \log N)^{1+\varepsilon}\) and \((N \log \log N)^{1/2}\) one can find a sequence \( \{n_k\} \) of integers such that the growth speed of \( D_N \{n_k x\} \) is \( \Psi(N) \) in the above sense. For all \( \gamma \in (0, 1/2] \), Aistleitner and Larcher [6] constructed an increasing sequence \( \{n_k\} \) of integers such that \( N D_N \{n_k x\} = O(N^\gamma) \) and \( N D_N \{n_k x\} = \Omega(N^\gamma - \varepsilon) \) a.e. for all \( \varepsilon > 0 \). They also constructed (see [7]) a sequence \( \{n_k\} \) with polynomial growth such that \( N D_N \{n_k x\} = O((\log N)^{2+\varepsilon}) \) a.e. for all \( \varepsilon > 0 \).

The main result of the present paper is the following

**Theorem 1.** Let \( \{\Psi(N)\} \) be a sequence of real numbers. Assume that there exists a constant \( N_0 \) such that

\[ 0 < \Psi(N) \leq \Psi(N + 1) \quad \text{for all } N \geq N_0, \]  

(4)

\[ \Psi(N) \geq (\log N)(\log \log N)^{1+\varepsilon} \quad \text{for some } \varepsilon > 0 \text{ and } N \geq N_0, \]  

(5)

\[ \Psi^2(N + 1) - \Psi^2(N) = o(\log \log \Psi^2(N)). \]  

(6)

Then for any \( \Sigma > 0 \), there exists a sequence \( \{n_k\} \) of positive integers satisfying \( 1 \leq n_{k+1} - n_k \leq 2 \) and

\[ \lim_{N \to \infty} \frac{N D_N \{n_k x\}}{\Psi(N)} = \Sigma \quad \text{a.e.} \]  

(7)
Note that for the function $\Psi^2(N) = N \log \log N$ we have

$$\Psi^2(N + 1) - \Psi^2(N) \sim \log \log \Psi^2(N)$$

and thus condition (6) means that the jumps of $\Psi^2(N)$ are of smaller order of magnitude than those of $N \log \log N$. Naturally, this implies that $\Psi^2(N) = o(N \log \log N)$ and thus the conditions of Theorem 1 bound the function $\Psi^2(N)$ between $(\log N)(\log \log N)^{1+\varepsilon}$ and $N \log \log N$ and require a certain smoothness of growth. Typical examples are $\Psi(N) = N^\alpha(\log N)^\beta(\log \log N)^\gamma$ where the parameters $\alpha, \beta, \gamma$ are chosen so that the order of growth of $\Psi^2(N)$ is between the previous bounds. Note that the theorem does not cover $\Psi(N) = (N \log \log N)^{1/2}$; the existence of $f_{n_k}g$ with (7) is already proved in [4] for $0 < \Sigma < \infty$, and in [2] for $\Sigma = \infty$. See also [9, 14].

As a related problem, we can ask if there exists a sequence $\{n_k\}$ such that $\sum_{k=1}^N \cos 2\pi n_k x$ grows with a given speed $\Psi(N)$. The law of the iterated logarithm by Erdős-Gál [11] states

$$\lim_{N \to \infty} \frac{1}{\sqrt{N \log \log N}} \sum_{k=1}^N \cos 2\pi n_k x = 1 \quad \text{a.e.}$$

(8)

for $\{n_k\}$ satisfying the Hadamard gap condition (2). As we will see in Section 4, for any $D > 0$ there exists an increasing $\{n_k\}$ such that (8) holds with the norming factor replaced by $c \sqrt{N (\log \log N)^D}$. The following theorem shows that any growth speed $O(\sqrt{N (\log \log N)^D})$ with small jumps is possible for $\sum_{k=1}^N \cos 2\pi n_k x$.

**Theorem 2.** Let $\{\Psi(N)\}$ be an sequence of real numbers. Assume that there exists a constant $N_0$ and $D > 0$ such that (4).

$$\Psi(N) \to \infty, \quad \text{and} \quad \Psi^2(N + 1) - \Psi^2(N) = o((\log \log \Psi^2(N))^{D}).$$

Then there exists a strictly increasing sequence $\{n_k\}$ of positive integers such that

$$\lim_{N \to \infty} \frac{1}{\Psi(N)} \sum_{k=1}^N \cos 2\pi n_k x = 1 \quad \text{a.e.}$$

(9)

In conclusion, we mention a number of open problems related to our results. Let $\mathcal{G}$ denote the class of functions $\Psi(N)$, $N = 1, 2, \ldots$ such that for some increasing sequence $\{n_k\}$ relation (7) holds for some constant $0 < \Sigma < \infty$. From Theorem 1 it follows that $\mathcal{G}$ contains all smoothly increasing functions $\Psi(N)$ with speed between $(\log N)(\log \log N)^{1+\varepsilon}$ for some $\varepsilon > 0$ and $(N \log \log N)^{1/2}$. By a classical result of W. Schmidt (see e.g. Kuipers and Niedereiter [22], p. 109) for any infinite sequence $\{x_k\}$ we have $ND_N\{x_k\} \geq c \log N$ for
ininitely many $N$ with an absolute constant $c$ and thus $\mathcal{G}$ contains no
functions $\Psi(N) = o(\log N)$. Hence assumption (5) in Theorem 1 is
nearly optimal; whether $\Psi(N) = (\log N)(\log \log N)^{\alpha}$, $0 \leq \alpha \leq 1$
belongs to $\mathcal{G}$ remains open. Concerning upper bounds for functions in $\mathcal{G}$,
the results of Baker [8] and Berkes and Philipp [9] imply that
\[
ND_N\{n_kx\} \leq \text{const} \cdot N^{1/2}(\log N)^{\gamma} \quad \text{a.e.}
\]
holds for all $\{n_k\}$ if $\gamma > 3/2$ but not if $\gamma \leq 1/2$. This implies that
for $\gamma > 3/2$ we have $N^{1/2}(\log N)^{\gamma} \notin \mathcal{G}$ and makes it plausible (but
does not prove) that $(N \log N)^{1/2} \in \mathcal{G}$. If this is true, condition (6) in
Theorem 1 can be replaced by
\[
\Psi^2(N + 1) - \Psi^2(N) = o(\log \Psi^2(N))
\]
allowing all smoothly growing functions $\Psi(N) = O(N \log N)^{1/2}$, an
essentially optimal result. Similar remarks hold for Theorem 2.

2. Key Proposition

We begin with proving a weaker version of Theorem 1.

**Proposition 3.** For any sequence $\{\psi(N)\}$ satisfying
\[
\psi(0) = 0, \quad \psi(N) \leq \psi(N + 1), \quad (10)
\]
\[
(\log N)(\log \log N)^{1+\varepsilon} = o(\psi(N)) \quad \text{for some} \quad \varepsilon > 0, \quad (11)
\]
\[
\psi^2(N + 1) - \psi^2(N) \leq \frac{1}{2}(4 \vee \log \log \psi^2(N)), \quad (12)
\]
there exists a sequence $\{n_k\}$ of positive integers satisfying $1 \leq n_{k+1} -
n_k \leq 2$ and
\[
\lim_{N \to \infty} \frac{ND_N\{n_kx\}}{\psi(N)} = \frac{\sqrt{2}}{4} \quad \text{a.e.} \quad (13)
\]
Set $G(x) = x/(4 \vee \log \log x)$, where $\log \log x$ is meant as $-\infty$ for
$x \leq 1$. Note that $G(x)$ is increasing. By (12), we can derive
\[
G(\psi^2(N + 1)) - G(\psi^2(N)) \leq \frac{\psi^2(N + 1) - \psi^2(N)}{4 \vee \log \log \psi^2(N)} \leq \frac{1}{2}. \quad (14)
\]
Let $\nu_i$ be the smallest $\nu$ satisfying $2i^3 \leq G(\psi^2(i^3 + \nu))$. Note that
$\nu_0 = 0$. By (14), we have
\[
G(\psi^2(i^3 + \nu_i)) = 2i^3 + e_i \quad \text{for some} \quad 0 \leq e_i < 1/2. \quad (15)
\]
Set $\Delta_i = \mathbb{N} \cap (2(i - 1)^3, 2i^3]$ and $\eta_i = 2i^3 - 2(i - 1)^3$. 

\[\]
By using (14), we have
\[ \eta_i - \frac{1}{2} \leq 2i^3 - 2(i - 1)^3 + e_i - e_{i-1} \]
\[ = G(\psi^2(i^3 + \eta_i)) - G((i - 1)^3 + \eta_{i-1}) \leq \frac{1}{2}(1 - \frac{1}{2} \eta_i + \nu_i - \nu_{i-1}) \].

By \( \eta_i \geq 2 \), we have
\[ \nu_i - \nu_{i-1} \geq (3/2)\eta_i - 1 \geq \eta_i \quad \text{and} \quad \nu_i \geq 2i^3. \quad (16) \]

Set \( \mu_k = 2\nu_i + 2(k - 2i^3) \) for \( k \in \Delta_i \). By \( \mu_{2i^3+1} = 2\nu_{i+1} - 2\eta_{i+1} + 2 \geq 2\nu_i + 2 \geq \mu_{2i^3} \), we see that \( \{\mu_k\} \) is strictly increasing.

We now introduce some notation. Denote by \( 1_{[a,b)} \) the indicator function of \([a, b)\), and put \( \bar{I}_{[a,b)}(x) = 1_{[a,b)}((x)) - (b - a) \). Then we have
\[ ND_N\{x_k\} = ND_N(x_1, \ldots, x_N) = \sup_{0 \leq a < b \leq 1} \left| \sum_{k=1}^{N} \bar{I}_{[a,b)}(x_k) \right|. \]

Put \( S = \{2^{-i} : l \in \mathbb{N}, i = 0, 1, \ldots, 2^{l} \} \), \( S^{2^i} = \{(a, b) : a, b \in S, a < b\} \), \( \phi_C(t) = \sqrt{Ct}(1 \sqrt{\log \log t}) \), and \( \sigma_{a,b} = \sqrt{(b - a)(1 - (b - a))} \). Let \( \{X_k\} \) be a sequence of independent random variables satisfying \( P(X_k = 1) = P(X_k = -1) = 1/2 \).

**Lemma 4.** We have
\[ \lim_{N \to \infty} \frac{1}{\phi_2(N)} \left| \sum_{k=1}^{N} \bar{I}_{[a,b)}(\mu_k x)X_k \right| = \sigma_{a,b} \quad (17) \]
for all \((a, b) \in S^{2^i}, \ a.e., \ a.s.\)

**Proof.** Since \( \mu_k \) is a strictly increasing sequence of integers, by Weyl’s theorem [27], \( \{\mu_k x\} \) is uniformly distributed modulo 1 a.e. Hence,
\[ B_N := \sum_{k=1}^{N} \bar{I}_{[a,b)}^2(\mu_k x) \sim N \int_{0}^{1} \bar{I}_{[a,b)}^2(y) \, dy = N \sigma_{a,b}^2 \to \infty \quad \text{a.e.} \]
if \( b - a \neq 0, 1 \). By Kolmogorov’s law of the iterated logarithm [21]
\[ \lim_{N \to \infty} \frac{1}{\phi_2(B_N)} \left| \sum_{k=1}^{N} \bar{I}_{[a,b)}(\mu_k x)X_k \right| = 1 \quad \text{a.s., a.e.,} \]
we see that (17) holds a.s., a.e. if \( 0 < b - a < 1 \). Clearly (17) holds if \( b - a = 0, 1 \). Since \( S^{2^i} \) is countable, we see that (17) holds for all \((a, b) \in S^{2^i}, \ a.s., \ a.e. \) By Fubini’s theorem, we have the conclusion. \( \square \)

**Lemma 5.** Suppose that \( l \in \mathbb{N} \) and \( 0 \leq i < 2^l \), we have
\[ \lim_{N \to \infty} \frac{1}{\phi_2(N)} \sup_{0 < c < 2^{-i}} \left| \sum_{k=1}^{N} \bar{I}_{([-2^{-i}, 2^{-i} + c]}(\mu_k x)X_k \right| \leq 4 \cdot 2^{-l/2} \quad \text{a.e., a.s.} \]
Proof. Denote \(1_{[a,b]}(\langle x \rangle)\) simply by \(1_{[a,b]}(x)\). By noting

\[
b_N = \sum_{k=1}^{N} 1_{[2^{-i},2^{-i-(i+1)})}(\mu_k x) \sim N \int_0^1 1_{(2^{-i},2^{-i-(i+1)})}(y) \, dy = N 2^{-l} \quad \text{a.e.}
\]

and by following the proof of Lemma 4 of [13], we can prove

\[
\lim_{N \to \infty} \frac{1}{\phi_2(N)} \sup_{0 < c < 2^{-l}} \left| \sum_{k=1}^{N} 1_{[2^{-i},2^{-i-(i+1)})}(\mu_k x) X_k \right| \leq \sqrt{10 \cdot 2^{-l}} \quad \text{a.e., a.s.}
\]

Thus together with the law of the iterated logarithm

\[
\lim_{N \to \infty} \sup \frac{c}{\phi_2(N)} \left| \sum_{k=1}^{N} X_k \right| = \lim_{N \to \infty} \frac{2^{-l}}{\phi_2(N)} \left| \sum_{k=1}^{N} X_k \right| \leq 2^{-l} \quad \text{a.s.,}
\]

we have the conclusion. \(\square\)

For \(0 \leq a < b \leq 1\), take \(l\) with \(b-a > 2^{-l}\) and take the largest \(i\) and \(j\) such that \(2^{-i} \leq a < 2^{-j} \leq b\). Then we have \(1_{[a,b]} = 1_{[2^{-i},2^{-j})} - 1_{[2^{-i},a]} + 1_{[2^{-j},b)}\) and \(\tilde{1}_{[a,b]} = 1_{[2^{-i},2^{-j})} - 1_{[2^{-i},a]} + 1_{[2^{-j},b)}\), which implies

\[
\max_{0 \leq i < j \leq 2^l} \lim_{N \to \infty} \frac{1}{\phi_2(N)} \left| \sum_{k=1}^{N} \tilde{1}_{[2^{-i},2^{-j})}(\mu_k x) X_k \right|
\]

\[
\leq \lim_{N \to \infty} \sup_{0 < a < b \leq 1} \frac{1}{\phi_2(N)} \left| \sum_{k=1}^{N} \tilde{1}_{[a,b]}(\mu_k x) X_k \right|
\]

\[
\leq \max_{0 \leq i < j \leq 2^l} \lim_{N \to \infty} \frac{1}{\phi_2(N)} \left| \sum_{k=1}^{N} \tilde{1}_{[2^{-i},2^{-j})}(\mu_k x) X_k \right|
\]

\[
+ 2 \max_{0 \leq i \leq 2^l} \lim_{N \to \infty} \sup_{0 < a \leq 2^{-l}} \frac{1}{\phi_2(N)} \left| \sum_{k=1}^{N} \tilde{1}_{[2^{-i},2^{-i+a})}(\mu_k x) X_k \right|
\]

By applying two lemmas above, we have

\[
\frac{1}{2} \leq \lim_{N \to \infty} \sup_{0 < a < b \leq 1} \frac{1}{\phi_2(N)} \left| \sum_{k=1}^{N} \tilde{1}_{[a,b]}(\mu_k x) X_k \right| \leq \frac{1}{2} + 8 \cdot 2^{-l/2} \quad \text{a.e., a.s.}
\]

which implies

\[
\lim_{N \to \infty} \sup_{0 < a < b \leq 1} \frac{1}{\phi_2(N)} \left| \sum_{k=1}^{N} \tilde{1}_{[a,b]}(\mu_k x) X_k \right| = \frac{1}{2} \quad \text{a.e., a.s.} \quad (18)
\]
By the relation $ND_N\{x_k + y\} = ND_N\{x_k\}$ and (1), we have

$$
\eta_i D_{\eta_i}(\mu_{2(i-1)^3+1}x, \mu_{2(i-1)^3+2}x, \ldots, \mu_{2i^3}x) = \eta_i D_{\eta_i}(2kx) = O((\log \eta_i)^2).
$$

Noting $ND_N\{\mu_k x\} \leq \sum_{i=1}^{j} \eta_i D_{\eta_i}(\mu_{2(i-1)^3+1}x, \mu_{2(i-1)^3+2}x, \ldots, \mu_{2i^3}x)$ for $N \in \Delta_j$, we have

$$
ND_N\{\mu_k x\} = O\left(\sum_{i=1}^{j} (\log \eta_i)^2\right) = O\left(N^{1/3}(\log N)^2\right) = o(\sqrt{N}) \quad \text{a.e.}
$$

by $j - 1 < (N/2)^{1/3}$. This together with (18) implies

$$
\lim_{N \to \infty} \sup_{0 \leq a < b \leq 1} \frac{1}{\phi_2(N)} \left| \sum_{k=1}^{N} \mathbb{I}_{[a,b]}(\mu_k x) \frac{X_k + 1}{2} \right| = \frac{1}{4} \quad \text{a.e., a.s.} \quad (19)
$$

Note that $\{\mu_k\}$ and $\{2k - 1\}$ are mutually disjoint. Let $\{\lambda_k\}$ be an arrangement in increasing order of $\{\mu_k\} \cup \{2k - 1\}$. By $\mu_{2i^3} = 2\nu_i$, we have $\#\{k : \mu_k \leq 2\nu_i\} = 2i^3$ and $\#\{k : 2k - 1 \leq 2\nu_i\} = \nu_i$, and thereby we have $\#\{k : \lambda_k \leq 2\nu_i\} = 2i^3 + \nu_i$ and $\lambda_{2i^3+\nu_i} = \nu_i$. We set

$$
Y_k = \begin{cases} 
1 & \lambda_k \notin 2\mathbb{N}, \\
(X_k + 1)/2 & \lambda_k \in 2\mathbb{N},
\end{cases}
$$

$I_N = \#\{k \leq N : \lambda_k \notin 2\mathbb{N}\}$, $J_N = \#\{k \leq N : Y_k = 1, \lambda_k \in 2\mathbb{N}\}$, and $H_N = \#\{k \leq N : Y_k = 1\} = I_N + J_N$. We have $I_{2i^3+\nu_i} = \#\{k \leq 2i^3+\nu_i : \lambda_k \notin 2\mathbb{N}\} = \#\{k : 2k - 1 \leq 2\nu_i\} = \nu_i$ and $H_{2i^3+\nu_i} = J_{2i^3+\nu_i} + \nu_i$. By the law of large numbers we have $J_{2i^3+\nu_i} \sim \frac{1}{2}\#\{k : \mu_k \leq 2\nu_i\} = i^3$ a.s. By (14), we have

$$
\left| G\left(\psi^2(H_{2i^3+\nu_i})\right) - G\left(\psi^2(i^3+\nu_i)\right) \right| \leq \frac{1}{2} |H_{2i^3+\nu_i} - (i^3+\nu_i)| = \frac{1}{2} |J_{2i^3+\nu_i} - i^3|.
$$

Dividing by $G(\psi^2(i^3 + \nu_i)) = 2i^2 + e_i$, we have

$$
\left| \frac{G(\psi^2(H_{2i^3+\nu_i}))}{2i^3 + e_i} - 1 \right| \leq \frac{1}{2} \frac{|J_{2i^3+\nu_i} - i^3|}{2i^3 + e_i} \rightarrow 0 \quad \text{a.s.}
$$

Therefore we have $G(\psi^2(H_{2i^3+\nu_i})) \sim 2i^3 + e_i \sim 2i^3 \sim 2J_{2i^3+\nu_i}$ a.s. Since $J_N$ and $H_N$ are increasing, for $N \in [ (i - 1)^3 + \nu_{i-1}, i^3 + \nu_i ]$ we have

$$
1 \sim \frac{G(\psi^2(H_{2(i-1)^3+\nu_{i-1}}))}{2J_{2i^3+\nu_i}} \leq \frac{G(\psi^2(H_N))}{2J_N} \leq \frac{G(\psi^2(H_{2i^3+\nu_i}))}{2J_{2(i-1)^3+\nu_{i-1}}} \sim 1,
$$

and thereby

$$
2J_N \sim G\left(\psi^2(H_N)\right) \quad \text{a.s.} \quad (20)
$$
By (1), we see \( ND_N \{(2k-1)x\} = O((\log N)(\log \log N)^{1+\epsilon/2}) \), which implies \( ND_N \{(2k-1)x\} = o((\log N)(\log \log N)^{1+\epsilon}) \) or
\[
\lim_{N \to \infty} \sup_{0 \leq a < b \leq 1} \frac{1}{A_N} \left| \sum_{k \leq N : \lambda_k \notin 2\mathbb{N}} \tilde{I}_{(a,b)}(\lambda_k x)Y_k \right| = 0 \quad \text{a.e., a.s.} \tag{21}
\]
for \( A_N = (\log I_N)(\log \log I_N)^\epsilon \). Since \( H_N \geq I_N \), it is valid for \( A_N = (\log H_N)(\log \log H_N)^\epsilon \). Because of (11), we see that (21) holds for \( A_N = \sqrt{2} \psi(H_N) \).

By (19), we have
\[
\lim_{N \to \infty} \sup_{0 \leq a < b \leq 1} \frac{1}{A_N} \left| \sum_{k \leq N : \lambda_k \in 2\mathbb{N}} \tilde{I}_{(a,b)}(\lambda_k x)Y_k \right| = \frac{1}{4} \quad \text{a.e., a.s.} \tag{22}
\]
for \( A_N = \phi_2(\#\{k \leq N : \lambda_k \in 2\mathbb{N}\}) \). By \( J_N \sim \frac{1}{2} \#\{k \leq N : \lambda_k \in 2\mathbb{N}\} \) a.s., we see that (22) is valid for \( A_N = \sqrt{2} \phi_2(J_N) \sim \phi_2(2J_N) \). (20) and \( \phi_2^2(G(\psi^2(N))) \sim 2\psi^2(N) \) imply \( \phi_2^2(J_N) \sim \phi_2^2(G(\psi^2(H_N)))/2 \sim \psi^2(H_N) \) a.s. Hence (22) holds for \( A_N = \sqrt{2} \psi(H_N) \). Combining these, we have
\[
\lim_{N \to \infty} \sup_{0 \leq a < b \leq 1} \frac{1}{A_N} \left| \sum_{k=1}^{N} \tilde{I}_{(a,b)}(\lambda_k x)Y_k \right| = \frac{1}{4} \quad \text{a.e., a.s.}
\]
Denoting by \( \{n_k\} \) the subsequence \( \{\lambda_k : Y_k = 1\} \), we have (13) a.s.

3. PROOF OF THEOREM 1

By (6), we have \( \Psi^2(N) = o(N \log \log \Psi^2(N)) \) and \( G(\Psi^2(N)) = o(N) \). For any \( C > 0 \), we see \( G(\phi_2^2(N)) \sim CN \) and hence \( G(\Psi^2(N)) \leq G(\phi_2^2(N)) \) or \( \Psi^2(N) \leq \phi_2^2(N) \) for large \( N \). Since it holds for any \( C > 0 \), we see that \( \Psi^2(N) = o(\phi_2^2(N)) \).

By (6), we can take \( N_1 > N_0 \) such that for all \( N \geq N_1 \),
\[
(2\sqrt{2} \Sigma \Psi(N+1))^2 - (2\sqrt{2} \Sigma \Psi(N))^2 \leq \frac{1}{2} \log \log(2\sqrt{2} \Sigma \Psi(N))^2. \tag{23}
\]
Take \( c \in \left(0, \frac{1}{2}\right) \) such that \( \phi_2^2(N_1) < (2\sqrt{2} \Sigma \Psi(N_1))^2 \) holds. We have \( (2\sqrt{2} \Sigma \Psi(N))^2 < \phi_2^2(N) \) for large \( N \geq N_1 \). Denote \( N_2 \) the minimum of such \( N \). Putting
\[
\psi(N) = \begin{cases} \phi_2(N) & N < N_2, \\ 2\sqrt{2} \Sigma \Psi(N) & N \geq N_2, \end{cases}
\]
it is clear that \( \psi(N) \) satisfies (10) and (11). As to the condition (12), we first prove it for \( \phi_2^2(N) \).

In the case \( \log \log(N+1) \geq 1 \), i.e. \( N \geq 15 \), we see \((N+1)(\log \log(N+1) - \log \log N) \leq ((N+1)/N) \log N \leq 2/\log 15 < \log \log 15 \leq \)
\[
\log \log N \text{ and } (N + 1) \log \log (N + 1) - N \log \log N < 2 \log \log N. \quad \text{If } c \log \log N \leq 1, \text{ then } 2c \log \log N \leq 2 \leq \frac{1}{2} (4 \vee \log \log \phi_c^2(N)). \quad \text{If } c \log \log N \geq 1, \text{ then } 2c \log \log N \leq \frac{1}{2} \log \log (cN \log \log N) \leq \frac{1}{2} (4 \vee \log \log \phi_c^2(N)). \quad \text{Therefore, when } \log \log (N + 1) \geq 1, \text{ we have } \phi_c^2(N + 1) - \phi_c^2(N) \leq 2c \log \log N \leq \frac{1}{2} (4 \vee \log \log \phi_c^2(N)).
\]

By \(\psi(N_2) - \psi^2(N_2 - 1) \leq (2\sqrt{2} \sum_{N} \Psi(N))^2 - \phi_c^2(N_2 - 1) \leq \phi_c^2(N_2) - \phi_c^2(N_2 - 1)\) together with (23), we conclude that \(\psi(N)\) satisfies (12).

Hence we can apply Proposition 3 to have the conclusion.

4. Proof of Theorem 2

Take an integer \(d \geq D \vee 2\) to satisfy
\[
\Psi^2(N + 1) - \Psi^2(N) = o(\log \log \Psi^2(N))^d). \quad (24)
\]

Put \(M_k = 2^{d-1}(k)\), \(L_k = \min\{n \mid \Psi^2(n) \geq (2^{d-1}/d)M_k(\log \log M_k)^d\}\), and \(L_k^+ = L_k + M_k + M_k + 1 - M_k\).

There exists \(K_\ast \) such that \(\max_{N \leq N_0} \Psi(N) < (2^{d-1}/d!)M_k(\log \log M_k)^d\) for all \(k \geq K_\ast\). From now on, we consider only for \(k \geq K_\ast\), for which we have \(L_k > N_0\).

By (24) and \(\Psi^2(L_k - 1) < (2^{d-1}/d!)M_k(\log \log M_k)^d\), we have
\[
(2^{d-1}/d!)M_k(\log \log M_k)^d \leq \Psi^2(L_k)
= o(\log \log \Psi^2(L_k - 1))^d + \Psi^2(L_k - 1)
\leq o((\log \log M_k(\log \log M_k)^d))^d) + (2^{d-1}/d!)M_k(\log \log M_k)^d,
\]
\[
\Psi^2(L_k)/(2^{d-1}/d!)M_k(\log \log M_k)^d \to 1, \log \log \Psi^2(L_k) - \log \log M_k \to 0 \quad \text{and } \log \log \Psi^2(L_k) \sim \log \log M_k \text{ in turn. Combining}
\]
\[
\Psi^2(L_{k+1}) - \Psi^2(L_k - 1)
\geq (2^{d-1}/d!)(M_{k+1}(\log \log M_{k+1})^d - M_k(\log \log M_k)^d)
\geq (2^{d-1}/d!)(M_{k+1} - M_k)(\log \log M_{k+1})^d
\]
and \(\Psi^2(L_{k+1}) - \Psi^2(L_k - 1) = (L_{k+1} - L_k + 1)o((\log \log \Psi^2(L_{k+1}))^d)\),
we have
\[
\frac{M_{k+1} - M_k}{L_{k+1} - L_k + 1} \leq \frac{o((\log \log \Psi^2(L_{k+1}))^d)}{(2^{d-1}/d!)(\log \log M_{k+1})^d} = o(1).
\]

Hence we see that there exists \(K_0\) such that
\[
L_{k+1} - L_k > M_{k+1} - M_k \quad \text{i.e., } L_{k+1} > L_k^+ \quad (k \geq K_0). \quad (25)
\]
By (24) we have $\Psi^2(N) \leq o(N (\log \log \Psi^2(N))^d)$, thereby $\log \Psi^2(N) < \log N + d \log \log \log \Psi^2(N)$, and $\log \Psi^2(N) \leq 2 \log N$ or $\Psi^2(N) \leq N^2$ for large $N$. Hence $\Psi^2(N) = o(N (\log \log N)^d)$. Hence we see $\Psi^2(M_k) = o(M_k (\log \log M_k)^d) = o(\Psi^2(L_k))$. It implies $M_k < L_k$ for large $k$. Take such $k \geq K_0$ and denote by $k_0$. We see $M_{k_0} < L_{k_0}$.

We define an non-decreasing sequence $\{a_k\}$ of positive integers as below. Put $a_1 = \cdots = a_k = 3$, take $a_{k_0} + 1$ large enough to satisfy

$$\gamma_{k_0 + 1} := \frac{1}{2} a_{k_0 + 1} \geq \frac{3}{2} a_{k_0} + (L_{k_0} - 1 - M_{k_0}) =: \gamma_{k_0 + 1}. \quad (26)$$

For $k \geq k_0$, inductively take $a_{k+2}$ large enough to satisfy $a_{k+2} \geq a_{k+1}$ and

$$\gamma_{k+2}^+ := \frac{1}{2} a_{k+2} \geq \frac{3}{2} a_{k+1} + (L_{k+1} - L_k) =: \gamma_{k+2}^+. \quad (27)$$

Put $\rho_j = a_j^j$. Since $\rho_j$ satisfies the Hadamard gap condition $\rho_{j+1}/\rho_j \geq a_{j+1} \geq 3$, by the law of the iterated logarithm we have

$$\lim_{N \to \infty} \frac{1}{\phi_1(N)} \sum_{j=1}^{N} \cos 2\pi \rho_j x = \lim_{N \to \infty} \frac{1}{\phi_1(N)} \left( \sum_{j=1}^{N} \cos 2\pi \rho_j x \right) = 1 \quad \text{a.e.} \quad (28)$$

From this, we drive

$$\lim_{N \to \infty} \frac{d!}{\phi_1(N)^d} \sum_{1 \leq m_1 < \ldots < m_d \leq N} \prod_{j=1}^{d} \cos 2\pi \rho_{m_j} x = 1 \quad \text{a.e.} \quad (29)$$

For a function $f(m_1, \ldots, m_d)$ on $\{1, \ldots, N\}^d$, define a signed measure $\nu$ on $\{1, \ldots, N\}^d$ by

$$\nu(A) = \sum_{(m_1, \ldots, m_d) \in A} f(m_1, \ldots, m_d) \quad (A \subset \{1, \ldots, N\}^d).$$

Let $J = \{(j, k) \mid 1 \leq j, k \leq N, \ j \neq k\}$. For $(j, k) \in J$, put $A_{(j,k)} = \{(m_1, \ldots, m_d) \in \{1, \ldots, N\}^d \mid m_j = m_k\}$.

Putting

$$f(m_1, \ldots, m_d) = \prod_{j=1}^{d} \cos 2\pi \rho_{m_j} x$$
and by applying the inclusion-exclusion principle
\[
\nu\left(\{1, \ldots, N\}^d \setminus \bigcup_{j \in J} A_j\right) = \nu(\{1, \ldots, N\}^d) - \sum_{j \in J} \nu(A_j) \\
+ \sum_{j_1, j_2 \in J: j_1 \neq j_2} \nu(A_{j_1} \cap A_{j_2}) - \cdots - \nu\left(\bigcap_{j \in J} A_j\right),
\]
we see that
\[
\left| \sum_{m_1, \ldots, m_d \leq N: m_j \neq m_k((j,k) \in J)} \prod_{j=1}^d \cos 2\pi \rho_{m_j} x - \left(\sum_{k=1}^N \cos 2\pi \rho_k x\right)^d \right|
\]
can be bounded by a linear combination of
\[
\prod_{j=1}^\beta \sum_{k=1}^N \cos^{\alpha_j} 2\pi \rho_k x \quad (\alpha_1 + \cdots + \alpha_\beta = d, \ \max_{j=1}^\beta \alpha_j \geq 2).
\]
Note that we can verify
\[
0 \leq \lim_{N \to \infty} \frac{1}{\phi_1(N)^d} \left| \prod_{j=1}^\beta \sum_{k=1}^N \cos^{\alpha_j} 2\pi \rho_k x \right| \\
\leq \prod_{j=1}^\beta \lim_{N \to \infty} \frac{1}{\phi_1(N)^\alpha_j} \left| \sum_{k=1}^N \cos^{\alpha_j} 2\pi \rho_k x \right| = 0 \ a.e.
\]
because
\[
\lim_{N \to \infty} \frac{1}{\phi_1(N)^\alpha} \left| \sum_{k=1}^N \cos^\alpha 2\pi \rho_k x \right| \leq \lim_{N \to \infty} \frac{N}{\phi_1(N)^\alpha} = 0
\]
holds for \( \alpha \geq 2 \). Hence by (28) we have
\[
\lim_{N \to \infty} \frac{1}{\phi_1(N)^d} \left| \sum_{m_1, \ldots, m_d \leq N: m_j \neq m_k((j,k) \in J)} \prod_{j=1}^d \cos 2\pi \rho_{m_j} x \right| \\
= \lim_{N \to \infty} \frac{1}{\phi_1(N)^d} \left(\sum_{k=1}^N \cos 2\pi \rho_k x\right)^d = 1 \ a.e.
\]
and thereby we see (29).

Let \( S_0 \) be a collection of \((b_1, b_2, \ldots) \in \{-1, 0, 1\}^N \) such that \( b_i = 0 \) for all large \( i \).

\textbf{Lemma 6.} The mapping \( S_0 \ni (b_1, b_2, \ldots) \mapsto \sum_{i=1}^\infty b_i a_i^j \in \mathbb{Z} \) is injective.
Proof. Because of $|\sum_{i=1}^{l-1} b_i a_i| \leq \sum_{i=1}^{l-1} a_i^l - \frac{1}{2} a_i^l$, we have

$$\sum_{i=1}^{l} b_i a_i^l \in \left( (b_i - \frac{1}{2}) a_i^l, (b_i + \frac{1}{2}) a_i^l \right),$$

and if $b_i \neq 0$, then

$$\sum_{i=1}^{l} b_i a_i^l \in (-\frac{3}{2} a_i^l, -\frac{1}{2} a_i^l) \cup (\frac{1}{2} a_i^l, \frac{3}{2} a_i^l) =: C_i.$$ (30)

Take $(b_1, b_2, \ldots) \in \mathcal{S}_0$ and $(b_1', b_2', \ldots) \in \mathcal{S}_0$ and assume $\sum_{i=1}^{\infty} b_i a_i^l = \sum_{i=1}^{\infty} b_i' a_i^l$. By putting $I = \max\{ i \mid b_i \neq 0 \}$ and $I' = \max\{ i \mid b_i' \neq 0 \}$, then we see that $\sum_{i=1}^{\infty} b_i a_i^l \in C_I$ and $\sum_{i=1}^{\infty} b_i a_i^l \in C_{I'}$. By $\frac{3}{2} a_i^l \leq \frac{1}{2} a_{i+1}^l$, we see that $C_I (I = 1, 2, \ldots)$ are mutually disjoint and $\max\{ i \mid b_i \neq 0 \} = \max\{ i \mid b_i' \neq 0 \}$. Because $\left( (b - \frac{1}{2}) a_i^l, (b + \frac{1}{2}) a_i^l \right) (b \in \mathbb{Z})$ are mutually disjoint, we see $b_I = b_{I'}$. Hence we have $\sum_{i=1}^{I-1} b_i a_i^l = \sum_{i=1}^{I-1} b_i' a_i^l$. In the same way, we can verify $b_i = b_i'$ for all $i < I$, and see that the mapping is injective.

By this lemma, we see that

$$\rho_{m_1} + \varepsilon_{d-1}\rho_{m_{d-1}} + \cdots + \varepsilon_1\rho_{m_1}$$ (31)

with $m_1 < m_2 < \cdots < m_d$ and $\varepsilon_1, \ldots, \varepsilon_d = \pm 1$ are all distinct. Denote by $\{l_i\}$ the arrangement in increasing order of this family.

Note that $M_k$ equals to the number of the sum of the type (31) with $m_1 < m_2 < \cdots < m_d \leq k$ and $\varepsilon_1, \ldots, \varepsilon_d = \pm 1$. By (30),

$$l_i \in \left( \frac{1}{2} a_N^l, \frac{3}{2} a_N^l \right), \quad (M_{N-1} < i \leq M_N).$$ (32)

Clearly

$$\prod_{j=1}^{d} \cos 2\pi \rho_{m_j} x = \frac{1}{2^{d-1}} \cos 2\pi (\rho_{m_d} + \varepsilon_{d-1}\rho_{m_{d-1}} + \cdots + \varepsilon_1\rho_{m_1}) x,$$

and

$$\sum_{1 \leq m_1 < \cdots < m_d \leq N} \prod_{j=1}^{d} \cos 2\pi \rho_{m_j} x = \frac{1}{2^{d-1}} \sum_{k=1}^{M_N} \cos 2\pi l_k x.$$

Hence by (29), we have

$$\lim_{N \to \infty} \frac{d!}{2^{d-1} \phi_1(N)^d} \sum_{k=1}^{M_N} \cos 2\pi l_k x = 1 \quad \text{a.e.}$$ (33)
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Put

$$B_N(x) = \max_{M_N+1 \leq Q \leq M_{N+1}} \left| \sum_{k=M_N+1}^{Q} \cos 2\pi l_k x \right|.$$  

By the Carleson-Hunt inequality [19] we have

$$\int_0^1 B_N^4(x) \, dx \leq C \int_0^1 \left( \sum_{k=M_N+1}^{M_{N+1}} \cos 2\pi l_k x \right)^4 \, dx$$

where $C$ is an absolute constant. Put

$$C_N(x) = \sum_{m_1, \ldots, m_d \leq N} \prod_{j=1}^{d} \cos 2\pi \rho_{m_j} x.$$  

By

$$\sum_{k=M_N+1}^{M_{N+1}} \cos 2\pi l_k x = 2^{d-1} \sum_{m_1 < \ldots < m_d \leq N} \prod_{j=1}^{d} \cos 2\pi \rho_{m_j} x$$

we have

$$\int_0^1 B_N^4(x) \, dx \leq C \left( \frac{2^{d-1}}{d!} \right)^4 \int_0^1 C_N^4(x)$$

As before, by the inclusion-exclusion principle, we see that $|C_N(x)|$ can be bounded from above by a linear combination of

$$\left| \prod_{j=1}^{\beta} \sum_{k=1}^{N-1} \cos^{\alpha_j} 2\pi \rho_k x \right| \quad (\alpha_1 + \cdots + \alpha_\beta = d - 1, \alpha_j \geq 1).$$

Put $S = \sum_{j=1}^{\beta} \alpha_j 1(\alpha_j > 1)$ and $T = \sum_{j=1}^{\beta} 1(\alpha_j = 1)$. $S + T = d - 1$ is clear. For $\alpha \geq 2$, we bound $\left| \sum_{k=1}^{N-1} \cos^{\alpha_j} 2\pi \rho_k x \right| \leq N \leq N^{\alpha/2}$ to have

$$\left| \prod_{j=1}^{\beta} \sum_{k=1}^{N-1} \cos^{\alpha_j} 2\pi \rho_k x \right| \leq N^{S/2} \sum_{k=1}^{N-1} \cos 2\pi \rho_k x.$$

By applying Theorem 8.20 of Zygmund [28], we have

$$\int_0^1 \left( \prod_{j=1}^{\beta} \sum_{k=1}^{N} \cos^{\alpha_j} 2\pi \rho_k x \right)^4 \, dx = O(N^{2S} N^{2T}) = O(N^{2(d-1)}).$$

Therefore we have

$$\int_0^1 B_N^4(x) \, dx = O(N^{2(d-1)}) \quad \text{and} \quad \sum_{N=1}^{\infty} \int_0^1 \left( \frac{B_N(x)}{N^{d/2}} \right)^4 \, dx < \infty.$$
By applying the Beppo-Levi Theorem we have $B_N = o(N^{d/2})$ a.e. By noting $M_N \sim N^{d-1/d!}$ and combining with (33), we have

$$\lim_{N \to \infty} \frac{1}{\sqrt{(2^{d-1/d})N(\log \log N)^d}} \sum_{i=1}^{N} \cos 2\pi l_i x = 1 \quad \text{a.e.} \quad (34)$$

Put

$$n_i = \begin{cases} l_i & \text{if } i \leq M_k, \\ l_{M_k} + (i - M_k) & \text{if } M_k < i < L_k, \\ l_{M_k+i+1-L_k} & \text{if } L_k \leq i < L^+_k, \\ n_{L^+_k} + (i + 1 - L^+_k) & \text{if } L^+_k \leq i < L_{k+1} \ (k \geq k_0), \end{cases}$$

We can verify that $\{n_k\}$ is strictly increasing. Actually by (32) and (26), we see

$$n_{L_k+1} = l_{M_k+1} > \gamma_{k+1}^+ \geq \gamma_{k+1}^- > l_{M_k} + (L_k - M_k) = n_{L_k} - 1,$$

and by (27) we see for $k \geq k_0$,

$$n_{L_{k+1}} = l_{M_{k+1}+1} > \gamma_{k+2}^+ \geq \gamma_{k+2}^- > l_{M_{k+1}} + (L_{k+1} - L^+_k) = n_{L_{k+1}} - 1.$$

Put $E = [1, M_k] \cup \bigcup_{k=0}^{\infty} [L_k, L^+_k)$, $F = \mathbb{N} \setminus E$, $E_N = E \cap [1, N]$, $F_N = F \cap [1, N]$, and $\eta_N = \#E_N$. By $\eta_{L_k} = M_k + 1$, we have $\Psi^2(L_k) \sim (2^{d-1/d})\eta_{L_k}(\log \log \eta_{L_k})^d$. By $\Psi^2(L_{k+1}) \sim \Psi^2(L_k)$, we have

$$\Psi^2(N) \sim (2^{d-1/d})\eta_N(\log \log \eta_N)^d \quad (35)$$

By (34), we see that

$$\lim_{N \to \infty} \frac{1}{A_N} \sum_{i \in E_N} \cos 2\pi n_i x = 1 \quad \text{a.e.}$$

holds for $A_N = \sqrt{(2^{d-1/d})\eta_N \log \log \eta_N}$, and by (35) we see that it holds for $A_N = \Psi(N)$.

If $N \in [L^+_k, L_k)$, we have $|\sum_{i=L_k+1}^{N} \cos 2\pi n_i x| \leq 2/|\sin \pi x|$. By $\Psi^2(N) \sim \eta_{L_k} \log \log \eta_{L_k} \sim (2^{d-1/d})k^d \log \log k$, we can see that

$$\max_{N \in [L^+_k, L_k)} \left| \sum_{i \in F_N} \cos 2\pi n_i x \right| \leq \frac{2k}{|\sin \pi x|} = o(\Psi(N)) \quad \text{a.e.}$$

Hence we can verify (9).
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