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GAP SERIES AND FUNCTION OF BOUNDED VARIATION

KATUSI FUKUYAMA (Kobe)

Abstract. We prove that the gap series \( \sum f(n_kx) \) does not behave like a series of independent random series when \( f \) is a function of bounded variation with rational discontinuity.

1. Introduction

This note mainly concerns with an asymptotic behavior of subsums of the series \( \sum f(n_kx) \), where \( \{n_k\} \) is a strictly increasing sequence of integers, and \( f \) is a locally square integrable real measurable function on \( \mathbb{R} \) with period 1 satisfying \( \int_0^1 f(x) \, dx = 0 \).

When \( n_k \) grows rapidly as \( n_{k+1}/n_k \to \infty \) and \( f \) is of \( \alpha \)-Lipschitz class \( (\alpha > 0) \), then \( \{f(n_kx)\} \) is almost independent and the classical law of the iterated logarithm holds (Takahashi [16]):

\[
\limsup_{N \to \infty} \frac{1}{\sqrt{N \log \log N}} \sum_{k=1}^N f(n_kx) = \left( 2 \int_0^1 |f(x)|^2 \, dx \right)^{1/2}, \quad \text{a.e.}
\]

The above gap condition is crucial and the left hand side may be non-constant (Erdős-Fortet, Cf. Kac [11, 12]) under Hadamard’s gap condition

\[ n_{k+1}/n_k \geq 1 + \rho > 1. \]

Only the upper half inequality

\[
\limsup_{N \to \infty} \frac{1}{\sqrt{N \log \log N}} \sum_{k=1}^N f(n_kx) \leq C_{\rho,f}, \quad \text{a.e.}
\]

holds for \( f \) of \( \alpha \)-Lipschitz class \( (\alpha > 0, \text{Takahashi [15]}) \) or of bounded variation (Philipp [14]). The inequality still holds under Takahashi’s gap condition

\[ n_{k+1}/n_k \geq 1 + c/k^\beta, \quad (c > 0, \beta < 1/2), \]

if \( f \) is of \( \alpha \)-Lipschitz class for some \( \alpha > 1/2 \) (Dhompongsa [5]).

Recent researches (Takahashi [17], Fukuyama[8, 9]) show the best possible bound

\[
\limsup_{N \to \infty} \frac{1}{\sqrt{N \log \log N}} \sum_{k=1}^N f(n_kx) \leq \|f\|_A = 2 \sum_{\nu=1}^\infty |\tilde{f}(\nu)|, \quad \text{a.e.}
\]

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under $\|f\|_A < \infty$ and Takahashi’s gap condition. By an obvious change of the proof, one can extend these results to the case of weighted sums:

$$\limsup_{N \to \infty} \frac{1}{\sqrt{N \log \log N}} \sum_{k=1}^{N} c_k f(n_k x) \leq \begin{cases} C_{\rho, f} \|f\|_A, & \text{a.e. if } c_k \in [-1, 1], \\ \frac{1}{2} C_{\rho, f}, & \text{else} \end{cases}$$

according to the cases of the Hadamard’s, resp. Takahashi’s gap condition. Since $\|f\|_A < \infty$ is an essential condition in the last result, it is worthwhile recalling the Bernstein’s theorem (Cf. Zygmund [Misc. Th. & Ex. 7 of Chap. VI of 18]), which claims $\|f\|_A < 1$ for $f \in \text{Lip}^{2\alpha} (\alpha > 1/2)$ or $\omega_2(f; \delta) = O(\delta^\alpha)$. Here $\omega_2(f; \delta)$ denotes the $L^2$ modulus of continuity, i.e., $\omega_2(f; \delta) = \sup_{|h| \leq \delta} \|f(\cdot + h) - f(\cdot)\|_2$.

We are interested in the case when the upper half inequality of law of the iterated logarithm fails to hold and the limsup is unbounded. Here we give a brief survey on the studies of this kind.

There exists a function $f \in \text{Lip}^{1/2}$ having the following property (Berkes [2]): for all $0 < \rho_k \to 0$, there exist $\{n_k\}$ with $n_{k+1}/n_k > 1 + \rho_k$ and $c_k = \pm 1$ such that

$$\limsup_{N \to \infty} \frac{1}{\sqrt{N \log \log N}} \sum_{k=1}^{N} c_k f(n_k x) = \infty, \quad \text{a.e.}$$

It explains the reason why the condition $\alpha > 1/2$ is necessary for the result of Dhompongsa [5]. Under weaker gap conditions than the Hadamard’s, stronger regularities on $f$ is needed to have the classical upper half inequality.

For functions of bounded variations, precise results are known (Berkes-Philipp [3, 4]): for all $0 < \rho_k \downarrow 0$ with $\rho_k/\rho_{2k} = O(1)$, there exists $\{n_k\}$ with $n_{k+1}/n_k > 1 + \rho_k$ such that

$$\limsup_{N \to \infty} \frac{1}{\log \log (1/\rho N) \sqrt{N \log \log N}} \sum_{k=1}^{N} \left( \langle n_k x \rangle - \frac{1}{2} \right) \geq C > 0, \quad \text{a.e.},$$

where $\langle x \rangle = x - \lfloor x \rfloor$. It explains the gap condition in the result of Philipp [14] is crucial and cannot be weakened. On the other hand, for function $f$ of bounded variation, non-classical upper half inequality

$$\limsup_{N \to \infty} \frac{1}{\log (1/\rho N^2) \sqrt{N \log \log N}} \sum_{k=1}^{N} f(n_k x) < \infty, \quad \text{a.e.}$$

holds.

Note that any function $f$ of bounded variation belongs to $\text{Lip}^2 1/2$. It is easily verified by $\hat{f}(\nu) = O(1/\nu)$ and Lemma 5 in section 2. Our next theorem claims that irregular behavior occurs for functions of wide subclass of $\text{Lip}^2 1/2$. 


Theorem. Let \( f \) be a locally square integrable real measurable function on \( \mathbb{R} \) with period 1 which satisfies \( \int_0^1 f(x) \, dx = 0 \) and \( f \in \text{Lip}^2 1/2 \).

(1) Suppose that there exists an \( x_0 \in \mathbb{Q} \) such that both limits \( f(x_0 + 0) \) and \( f(x_0 - 0) \) exist and are distinct. Then for all \( 0 < \rho_k \downarrow 0 \) and \( c \in (12/25, 1/2) \) there exist \( \{n_k\} \) with \( n_{k+1}/n_k > 1 + \rho_k \), \( \{c_k\} \) with \( |c_k| \leq 1 \) and standard normal i.i.d. \( \{G_k\} \) on \( ([0, 1), B, dx) \) such that setting \( S_N(x) = \sum_{k=1}^N c_k f(n_k x) \), \( v_N = \int_0^1 S_N^2(x) \, dx \) and \( W_N = G_1 + \cdots + G_N \) we have

\[
v_N/N \to \infty, \quad v_{N-1} \sim v_N \quad \text{and} \quad S_N(x) = W_{[v_N]}(x) + O(v_N^c) \quad \text{a.e. \((N \to \infty)\).}
\]

(2) Suppose that \( \sum |\hat{f}(\nu)| = \infty \), and that \( \{\text{Re} \, \hat{f}(\nu)\} \) and \( \{\text{Im} \, \hat{f}(\nu)\} \) are negative, positive, or alternating. Then the conclusion of (1) holds with \( c_k \equiv 1 \).

It is known (Cf. eg. [Th. 4.7.2.; Itô 10]) that \( v_N \to \infty \) and \( v_{N-1}/v_N \to 1 \) implies \( \limsup_{N \to \infty} W_{[v_N]}/\sqrt{2v_N \log \log v_N} = 1 \), a.e. Hence (1) yields the law of the iterated logarithm:

\[
\limsup_{N \to \infty} \frac{1}{\sqrt{2v_N \log \log v_N}} \sum_{k=1}^N c_k f(n_k x) = 1, \quad \text{a.e.}
\]

By \( v_N/N \to \infty \), we see that dependence among \( \{f(n_k x)\} \) can not be neglected and gives irregular growth of the variance. This mechanism explains how limit theorems for gap sequence break by relaxing the Hadamard’s gap condition.

In (2) we assume some structure of Fourier coefficients and drop the need of coefficients \( c_k \). Although the conditions seems artificial, the result is sharp and we have the following important examples: if we apply it to functions

\[
\log |2 \sin \pi x| = -\sum_{\nu=1}^\infty \frac{\cos 2\pi \nu}{\nu}, \quad \log |2 \cos \pi x| = \sum_{\nu=1}^\infty (-1)^{\nu-1} \frac{\cos 2\pi \nu}{\nu} \in \text{Lip}^2 1/2,
\]

we can construct an \( \{n_k\} \) with arbitrarily weaker gaps than the Hadamard’s which obeys the following asymptotics:

\[
\left| \prod_{k=1}^N 2 \sin \pi n_k x \right| = e^{W_{v_N} + o(v_N^c)} \quad \text{and} \quad \left| \prod_{k=1}^N 2 \cos \pi n_k x \right| = e^{W_{v_N} + o(v_N^c)}, \quad \text{a.e.}
\]

for some \( c \in (12/25, 1/2) \) and \( v_N \) with \( v_N/N \to \infty \). These results have the contrast with the following results for sum (Berkes [1]), in which the dependence among the sequence does not reflect in the asymptotics: \( \sum_{k=1}^N \sqrt{2} \sin \pi n_k x = W_N + o(N^c) \) and \( \sum_{k=1}^N \sqrt{2} \cos \pi n_k x = W_N + o(N^c) \) for \( n_k \) satisfying Takahashi’s gap condition.

Another important example is \( \sum_{\nu=1}^\infty (\sin 2\pi \nu x)/\nu \log \nu \), which is absolutely continuous, and hence if of bounded variation. Even such good regularity cannot assure the classical law of the iterated logarithm.
2. Control of variance

In this and next sections, we give a proof of part (1) of Theorem. We divide the proof into several lemmas.

Let \( p_1, p_2, \ldots \) be a sequence of whole prime numbers and suppose that \( f \) is discontinuous at \( x_0 = q/(p_1^{j_1} \ldots p_r^{j_r}) \) where \( q \) is relatively prime with \( p_1^{j_1} \ldots p_r^{j_r} \). Denote \( N_0 = \mathbb{N} \cup \{0\}, B_{\tau', \tau} = \{p_{\tau'}^{j_1} \ldots p_{\tau}^{j_r} \mid j_1, \ldots, j_r \in N_0\}, \)
\( \mathcal{P}_{\tau', \tau} = \{n \in \mathbb{Z}^* \mid \gcd(n, p_{\tau'} \ldots p_{\tau}) = 1\} \), and \( C = \{z \in \mathbb{C} \mid |z| = 1\} \).

Next two lemmas are contain main ideas of this paper.

**Lemma 1.** If \( \tau' \leq \tau, f \in \text{Lip}^2 \varepsilon \) for some \( \varepsilon > 0 \), and \( \int_0^1 f(x) \, dx = 0 \), then
\[
\sum_{n,m \in B_{\tau', \tau}, \varepsilon: \gcd(n,m)=1} \int_0^1 |f(nx)f(mx)| \, dx < \infty.
\]

Proof: It is enough to prove assuming \( \varepsilon = 1 \). The proof is by probabilistic method. Let \( \{Y_{p_k,j} \mid k, j \in \mathbb{N}\} \) be a system of \( C \)-valued independent random variables with \( EY_{p_k,j} = p_k^{-H} e_k \). By putting \( X_{p_k,j} = Y_{p_k,1} \ldots Y_{p_k,j} \) and \( X_{p_k,0} = 1 \), we prove that
\[
\sigma_{\tau, \tau} := \lim_{J \to \infty} \frac{1}{J^\tau} \int_0^1 \left| \sum_{j_1, \ldots, j_\tau=0}^{J-1} X_{p_1,j_1} \ldots X_{p_\tau,j_\tau} f(p_1^{j_1} \ldots p_\tau^{j_\tau} x) \right|^2 \, dx
\]
is greater than the right hand side of our inequality, a.s. Put \( n^\pm = \max\{0, \pm n\} \), and \( \delta_{<m} = 1 \) if \( n < m \) and \( \delta_{<m} = 0 \) otherwise. It holds that
\[
A_{j,j}^{(s)} := \frac{1}{J} \sum_{l=0}^{J-|j|-1} X_{p_s,j+l}^* X_{p_s,j-l}^* p_s^{-H|j|} e_s^j \quad (J \to \infty, \ s \in \mathbb{N}, \ j \in \mathbb{Z}), \text{ a.s.}
\]
Actually, when \( j \geq 0 \), the summand equals to \( Y_{p_s,l+1} \ldots Y_{p_s,l+j} \), which forms \((j-1)\)-dependent sequence of bounded random variables and hence the law of
the large numbers yields the relation above. By taking the complex conjugate, we have $A_{j,l}^{(s)} = (A_{j,-l}^{(s)})^*$ and can verify the case $j < 0$.

Let us put $I_j^{j_1, \ldots, j_\tau} = \int_1^f (p_1^{j_1} \ldots p_\tau^{j_\tau}) \times (p_1^{j'_1} \ldots p_\tau^{j'_\tau}) \times dx$. By applying formal change of variables $\sum_{k,k'} \xi_{k,k'} = \sum_{j \in \mathbb{Z}} \sum_{l \in \mathbb{N}_0} \xi_{j+l,j-l}^{+} + i$ to expansion

$$LV = \lim_{J \to \infty} \frac{1}{J} \sum_{k_1, k'_1, \ldots, k_\tau, k'_\tau \in \mathbb{N}_0} \prod_{s=1}^{\tau} X_{p_s}, k_s, X_{p_s, k'_s}^{+} \delta_{<j}^{+} \delta_{<j}^{-}.$$ 

Lemma 1 claims $\sum_{j_1, \ldots, j_\tau} |I_{j_1}^{j_1, \ldots, j_\tau}| < \infty$, which justifies the above change of order of summations. By noting this inequality and $|A_{j_1, \ldots, j_\tau}| \leq 1$, we can take the termwise limit (dominated convergence theorem for series) and have

$$LV = \sum_{j_1, \ldots, j_\tau} \mathbb{Z} p_1^{-H[j_1]} \ldots p_\tau^{-H[j_\tau]} e_1^{j_1} \ldots e_\tau^{j_\tau} I_{j_1}^{j_1, \ldots, j_\tau} = \lim_{J \to \infty} \sum_{j_1, \ldots, j_\tau} A_{j_1, j_\tau}^{(1)} \ldots A_{j_1, \tau}^{(\tau)} I_{j_1}^{j_1, \ldots, j_\tau}.$$ 

In the case $\gcd(n, m) = 1$, by absolute convergence of the series, we have

$$\lim_{J \to \infty} \frac{1}{J} = \sum_{k_1, k'_1, \ldots, k_\tau, k'_\tau \in \mathbb{N}_0} \prod_{s=1}^{\tau} X_{p_s}, k_s, X_{p_s, k'_s}^{+} \delta_{<j}^{+} \delta_{<j}^{-}.$$

This together with $p_1^{-H[j_s]} e_1^{j_s} = EX_{p_s, j_s}^{+} X_{p_s, j_s}^{-}$ implies

$$LV = \sum_{j_1, \ldots, j_\tau} \mathbb{Z} e_1^{j_1} \ldots e_\tau^{j_\tau} X_{p_s, j_s}^{+} X_{p_s, j_s}^{-}.$$ 

Thanks to $\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} \xi_{j+k,j-k} = \sum_{j,j' \in \mathbb{N}_0} \xi_{j,j'}$, one can verify

$$LV = \sum_{j_1, \ldots, j_\tau} E \left( \prod_{s=1}^{\tau} X_{p_s, j_s}^{+} X_{p_s, j_s}^{-} \right) \lim_{J \to \infty} \left( \sum_{j_1, \ldots, j_\tau} \mathbb{Z} e_1^{j_1} \ldots e_\tau^{j_\tau} X_{p_s, j_s}^{+} X_{p_s, j_s}^{-} \right)^2.$$ 

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Note that $|\hat{f}(\nu)| = O(\nu^{-\varepsilon})$ and that the series inside the last expectation is absolutely and uniformly convergent. Hence one can justify the above change of order of summations. By applying Jensen inequality, we can achieve our goal

$$LV \geq \sum_{r \in P_{1,\tau}} \left| E \left( \sum_{j_1, \ldots, j_{\tau} \in \mathbb{N}_0} X_{p_{j_1}, j_1} \ldots X_{p_{\tau}, j_{\tau}} \frac{1}{\phi(m)} \frac{p_{j_1} \ldots p_{j_{\tau}}}{m} f(p_{j_1} \ldots p_{j_{\tau}}) \right) \right|^2$$

$$= \sum_{r \in P_{1,\tau}} \sum_{j_1, \ldots, j_{\tau} \in \mathbb{N}_0} \left| \frac{1}{\phi(m)} \frac{p_{j_1} \ldots p_{j_{\tau}}}{m} f(p_{j_1} \ldots p_{j_{\tau}}) \right|^2 .$$

**Lemma 3.** Let $f \in L^1$, $H > 0$, $m \in P_{\tau, \tau}$, and assume $\gcd(m, q) = 1$. Denote the Euler function by $\varphi$. Then there exist $e_{\tau}, \ldots, e_{\tau} \in C$ with

$$\left| \sum_{j_1, \ldots, j_{\tau} \in \mathbb{N}_0} (p_{j_1} \ldots p_{j_{\tau}})^{-H} e_{j_1} \ldots e_{j_{\tau}} \frac{1}{\phi(m)} \frac{p_{j_1} \ldots p_{j_{\tau}}}{m} f(p_{j_1} \ldots p_{j_{\tau}}) \right| \geq \frac{1}{\sqrt{\varphi(m)}} \left| \sum_{j_1, \ldots, j_{\tau} \in \mathbb{N}_0} (p_{j_1} \ldots p_{j_{\tau}})^{-H} \exp(2\pi i \frac{p_{j_1} \ldots p_{j_{\tau}} q}{m}) f(p_{j_1} \ldots p_{j_{\tau}}) \right| .$$

**Proof:** Since $|\hat{f}(\nu)| \leq 1$, the above two series are absolutely convergent. Put $G = (\mathbb{Z}/m\mathbb{Z})^*$ and $\hat{G}$ be its dual group. They are isomorphic and having order $\phi(m)$. Let $\Pi$ be a natural projection from $\mathbb{Z}$ over $\mathbb{Z}/m\mathbb{Z}$ and put $\bar{\chi}(n) = \chi(\Pi(n))$. Then $\bar{\chi}(p_{j_1} \ldots p_{j_{\tau}}) = \bar{\chi}(p_{j_1}) \ldots \bar{\chi}(p_{j_{\tau}})$. Put $h(g) = \exp(2\pi i g q/m)$ and $a_g^* = \sum_{j_1, \ldots, j_{\tau} \in \mathbb{N}_0} (p_{j_1} \ldots p_{j_{\tau}})^{-H} \frac{1}{\phi(m)} \frac{p_{j_1} \ldots p_{j_{\tau}}}{m} f(p_{j_1} \ldots p_{j_{\tau}})$. Then we have the following two fundamental formulas:

$$\sum_{j_1, \ldots, j_{\tau} \in \mathbb{N}_0} (p_{j_1} \ldots p_{j_{\tau}})^{-H} \frac{1}{\phi(m)} \frac{p_{j_1} \ldots p_{j_{\tau}}}{m} f(p_{j_1} \ldots p_{j_{\tau}}) = \sum_{g \in G} h(g) a_g^* ,$$

$$\sum_{j_1, \ldots, j_{\tau} \in \mathbb{N}_0} (p_{j_1} \ldots p_{j_{\tau}})^{-H} \bar{\chi}(p_{j_1}) \ldots \bar{\chi}(p_{j_{\tau}}) \frac{1}{\phi(m)} \frac{p_{j_1} \ldots p_{j_{\tau}}}{m} f(p_{j_1} \ldots p_{j_{\tau}}) = \sum_{g \in G} \chi(g) a_g^* ,$$

Thus the assertion follows from $\max_{\chi \in \hat{G}} |\sum_{g \in G} \chi(g) a_g^* | \geq |\sum_{g \in G} h(g) a_g^* | / \sqrt{\phi}$. Since $\{\chi\}_{\chi \in \hat{G}}$ forms an orthogonal base of the space of $C$-valued functions on $G$ (Cf. [pp. 218 of 6]), by putting $\hat{h}(\chi) = \sum_{g \in G} h(g) \chi^*(g) / \phi$, we have $h(g) = \sum_{\chi \in \hat{G}} \hat{h}(\chi) \chi(g)$ and $\sum_{\chi \in \hat{G}} |\hat{h}(\chi)|^2 = \sum_{g \in G} |h(g)|^2 / \phi = 1$. By noting $\sum_{\chi \in \hat{G}} |\hat{h}(\chi)| \leq \left( \sum_{\chi \in \hat{G}} |\hat{h}(\chi)|^2 \right)^{1/2} \left( \sum_{\chi \in \hat{G}} 1 \right)^{1/2} = \sqrt{\phi}$ and

$$\left| \sum_{g \in G} h(g) a_g^* \right| = \left| \sum_{\chi \in \hat{G}} \hat{h}(\chi) \sum_{g \in G} \chi(g) a_g^* \right| \leq \sum_{\chi \in \hat{G}} |\hat{h}(\chi)| \max_{\chi \in \hat{G}} \left| \sum_{g \in G} \chi(g) a_g^* \right| ,$$

we complete the proof. ■
Lemma 4. If \( s_n = x_1 + \cdots + x_n \to \infty \), then \( \lim_{H \uparrow 0} \liminf_{N \to \infty} \sum_{n=1}^{N} n^{-H} x_n = \infty \).

Proof. Put \( s_0 = 0 \). Because of \( s_n \to \infty \), the sequence \( t_n = \min_{k \geq n} s_k \) satisfies \( t_n \uparrow \infty \) and \( -\infty < t_n \leq s_n \). By applying Abel transform twice, we have \( \sum_{n=1}^{N} n^{-H} x_n = \sum_{n=1}^{N} s_n (n^{-H} - (n + 1)^{-H}) + s_N (N + 1)^{-H} \geq \sum_{n=1}^{N} t_n (n^{-H} - (n + 1)^{-H}) + t_N (N + 1)^{-H} = \sum_{n=1}^{N} n^{-H} (t_n - t_{n-1}) + t_0 \).

Therefore, \( \liminf_{N \to \infty} \sum_{n=1}^{N} n^{-H} x_n \geq \lim_{n \to \infty} n^{-H} (t_n - t_{n-1}) + t_0 \). Since the left hand side diverges to infinity as \( H \downarrow 0 \), we have the conclusion. \( \blacksquare \)

Lemma 5. (1) If \( \hat{f}(\nu) = O(1/\nu) \), then \( f \in \text{Lip}^2 1/2 \).

(2) If \( f \in \text{Lip}^2 1/2 \), then \( \sum_{\nu=1}^{\infty} \nu^{-H} |\hat{f}(\nu)| < \infty \) \((H > 0)\) and \( f \in L^p \) \((p \geq 1)\).

Proof: (1) It follows from \( \|f(x+h) - f(x-h)\|^2 = \sum_j |\hat{f}(\nu)|^2 \sin^2 \pi \nu h = O(\sum_{\nu>1/\pi h} (\pi h)^2 + \sum_{\nu \geq 1/\pi h} 1/\nu^2) = O(h) \).

(2) \( f \in \text{Lip}^2 1/2 \) implies \( \{\hat{f}(\nu)\} \in l^q \) \((q > 1)\). (Cf. Zygmund [Misc. Th. & Ex. 7 of Chap. VI of 18]). Let \( H \leq 1 \). Then \( \nu^{-H} \in l^{2/H} \), and hence by letting \( q \) be the dual of \( 2/H \) and noting \( \{\hat{f}(\nu)\} \in l^q \), we have \( \sum \nu^{-H} |\hat{f}(\nu)| < \infty \).

If \( p > 2 \), by putting \( q \) be its dual and noting \( \{\hat{f}(\nu)\} \in l^q \), by Hausdorff-Young theorem, we have \( f \in L^p \) for \( p > 2 \). The case \( p \leq 2 \) is clear. \( \blacksquare \)

Now we are in a position to state our key lemma to control the behavior of variance.

Lemma 6. Assume the conditions of part (1) of Theorem. For any \( L > 0 \), we can take \( \tau' < \tau \), and \( c_{j_{\tau}', \ldots, j_{\tau}} \in [-1, 1] \) satisfying

\[
\limsup_{j \to \infty} \frac{1}{J_{\tau'-1}+1} \int_0^1 \left| \sum_{j_{\tau}', \ldots, j_{\tau}=0}^{J-1} c_{j_{\tau}', \ldots, j_{\tau}} f(p_{j_{\tau}'}^{j_{\tau}'} \cdots p_{j_{\tau}}^{j_{\tau}} x) \right|^2 dx \geq L.
\]

Proof: Let \( \tilde{S}_N(x) \) be the \( N\)-th subsums of conjugate Fourier series of \( f \), i.e., \( \tilde{S}_N(x) = \sum_{\nu=1}^{N} 2 \text{Im} \hat{f}(\nu) \exp(2\pi i \nu x) \). By the discontinuity at \( x_0 \), we have \( \tilde{S}_N(x_0) \to \infty \) or \( \tilde{S}_N(x_0) \to -\infty \) (Cf. Zygmund [Chap. II 8 of 18]). By applying previous two lemmas, we have \( \lim_{H \uparrow 0} \limsup_{N \to \infty} \nu^{-H} \text{Im} \hat{f}(\nu) \exp(2\pi i \nu x_0) = \pm \infty \).

Since the series is absolutely convergent, and since \( B_{1,\tau} \uparrow \mathbb{N} (\tau \to \infty) \), we can take \( \tau \in \mathbb{N} \) and \( H > 0 \) such that \( |\Delta| \geq |\sum_{\nu \in B_{1,\tau}} \nu^{-H} \text{Im} \hat{f}(\nu) \exp(2\pi i \nu x_0)| \geq L \), where \( \Delta = \sum_{j_1, \ldots, j_\sigma} \left( p_{j_1}^{j_1} \cdots p_{j_\tau}^{j_\tau} \right)^{-H} \exp \left( \frac{2\pi i p_{j_1}^{j_1} \cdots p_{j_\tau}^{j_\tau} q}{p_{j_1}^{j_1} \cdots p_{j_\sigma}^{j_\sigma}} \right) \hat{f}(p_{j_1}^{j_1} \cdots p_{j_\tau}^{j_\tau}) \hat{f}(p_{j_1}^{j_1} \cdots p_{j_\tau}^{j_\tau}) \). For \( (j_1, \ldots, j_\sigma) \in \Lambda = (N_0^\sigma \cap [0, J_1] \times \cdots \times [0, J_\sigma]) \setminus \{(J_1, \ldots, J_\sigma)\} \), put

\[
\Xi_{j_1, \ldots, j_\sigma} = \sum_{j_{\sigma+1}, \ldots, j_{\tau} \in \mathbb{N}_0} \left( p_{j_{\sigma+1}}^{j_{\sigma+1}} \cdots p_{j_{\tau}}^{j_{\tau}} \right)^{-H} \exp \left( \frac{2\pi i p_{j_{\sigma+1}}^{j_{\sigma+1}} \cdots p_{j_{\tau}}^{j_{\tau}} q}{p_{j_{\sigma+1}}^{j_{\sigma+1}} \cdots p_{j_{\tau}}^{j_{\tau}} \hat{f}(p_{j_{\sigma+1}}^{j_{\sigma+1}} \cdots p_{j_{\tau}}^{j_{\tau}}) \hat{f}(p_{j_{\sigma+1}}^{j_{\sigma+1}} \cdots p_{j_{\tau}}^{j_{\tau}})},
\]
\[ \Upsilon_{j_1, \ldots, j_\sigma} = \sum_{j_{\sigma+1}, \ldots, j_{\tau} \in \mathbb{N}_0} (p_{j_{\sigma+1}}^{j_1} \ldots p_{j_{\tau}}^{j_\sigma}) - H \hat{f}(p_1^{j_1} \ldots p_{j_{\tau}}^{j_\tau}), \]

\[ \Upsilon = \sum_{j_1, \ldots, j_{\tau} \in \mathbb{N}_0} (p_1^{j_1} \ldots p_{j_{\tau}}^{j_\tau}) - H \hat{f}(p_1^{j_1} \ldots p_{j_{\tau}}^{j_\tau}). \]

Then we have \( \Delta = \Upsilon + \sum_{(j_1, \ldots, j_\sigma) \in \Lambda} (p_1^{j_1} \ldots p_{j_{\tau}}^{j_\sigma}) - H(\Xi_{j_1, \ldots, j_\sigma} - \Upsilon_{j_1, \ldots, j_\sigma}). \) The inequality \( |\Delta| \geq L \) implies that at least one of \( |\Upsilon|, |\Upsilon_{j_1, \ldots, j_\sigma}|, \) and \( |\Xi_{j_1, \ldots, j_\sigma}| \) \((j_1, \ldots, j_\sigma) \in \Lambda)\) is greater than \( L/2J_1 \ldots J_\sigma. \)

In case when \( |\Xi_{j_1, \ldots, j_\sigma}| \geq L/2J_1 \ldots J_\sigma, \) by applying Lemma 3 with \( r = \mu_1^{j_1} \ldots \mu_{j_{\tau}}^{j_\sigma}, \) we have

\[ \left| \sum_{j_{\sigma+1}, \ldots, j_{\tau} \in \mathbb{N}_0} (p_{j_{\sigma+1}}^{j_1} \ldots p_{j_{\tau}}^{j_\sigma}) - H e_{j_{\sigma+1}}^{j_1} \ldots e_{j_{\tau}}^{j_\sigma} \hat{f}(p_1^{j_1} \ldots p_{j_{\tau}}^{j_\tau}) \right| \geq L' \]

for some \( e_{\sigma+1}, \ldots, e_{\tau} \in C, \) where \( L' = L/2J_1 \ldots J_\sigma \varphi(p_1^{j_1} \ldots p_{j_{\tau}}^{j_\sigma})^{1/2}. \) In case when \( |\Upsilon_{j_1, \ldots, j_\sigma}| \geq L/2J_1 \ldots J_\sigma, \) the same estimate clearly holds for \( e_{\sigma+1} = \cdots = e_{\tau} = 1. \) In case when \( |\Upsilon| \geq L/2J_1 \ldots J_\sigma, \) replacing \( \sigma + 1 \) by 1, the same estimate again holds for \( e_1 = \cdots = e_{\tau} = 1. \)

By applying Lemma 2 to this estimate, we can take \( \tau', \tau, \) and \( d_{j_{\tau'}, \ldots, j_{\tau}} \in C \) such that \( \lim_{J \to \infty} \frac{1}{J_{\tau'} - J_{\tau} + 1} \int_0^1 \left| \sum_{j_{\tau'}, \ldots, j_{\tau} = 0}^{J-1} d_{j_{\tau'}, \ldots, j_{\tau}} f(p_{j_{\tau'}}^{j_1} \ldots p_{j_{\tau}}^{j_\tau} x) \right|^2 dx \geq L'. \)

By considering the real or imaginary part, we complete the proof. \( \blacksquare \)

3. Almost sure invariance principles

By the existence of \( f(x_0 + 0) \neq f(x_0 - 0), \) we see that \( f \) cannot be equal to 0 a.e., and hence we have \( \|f\|_2 > 0. \) For given \( c \in (12/25, 1/2), \) put \( d \) by \( 1/2 - (1-d)/50 = c. \) We have \( d \in (0, 1). \) Since \( (p(1/2 - (1-d)/50) - 1)/(1+p) \to 1/2 - (1-d)/50 > d/4 \) as \( p \to \infty, \) we can take large enough \( p \) such that

\[ pd/4 > 1 \quad \text{and} \quad -1 > d(1+p)/4 - p(1/2 - (1-d)/50). \]

By Lemma 5, we have \( \|f\|_p < \infty. \) By Lemma 6, for all \( L \in \mathbb{N}, \) we can take \( K_L \in \mathbb{N}, \) \( m_1^{(L)} < \cdots < m_{K_L}^{(L)} \), and \( c_1^{(L)}, \ldots, c_{K_L}^{(L)} \in [-1, 1] \) such that

\[ \frac{1}{K_L} \int_0^1 |Z_L(x)|^2 dx \geq L \|f\|_2^2, \quad \text{where} \quad Z_L(x) = \sum_{j=1}^{K_L} c_j^{(L)} f(m_j^{(L)} x). \]

Especially, we take \( K_1 = 1, \) \( c_1^{(1)} = 1, \) \( m_1^{(1)} = 1, \) and \( Z_1(x) = f(x), \) which obviously satisfies the inequality above.

Let us put \( \pi_1 = \rho_1 \) and take \( \pi_L > 0 \) such that \( m_{K_{k+1}}^{(L)} / m_k^{(L)} > 1 + \pi_L. \) \((1 \leq k < K_L). \) We may assume that \( \pi_1 > \pi_2 > \cdots > 0 \) holds. For all \( L, \) \( \rho_k \leq \pi_L \) holds for large enough \( k, \) because of \( \pi_L > 0 \) and \( \rho_k \downarrow 0. \)
Let us construct \( L(I) \) inductively as follows: First, put \( L(1) = 1 \). After having constructed \( L(1), \ldots, L(I - 1) \), let us denote \( L(I) \) the largest \( L \) satisfying the following two inequalities:

\[
\rho K_{L(1)} + \cdots + K_{L(I-1)+1} \leq \pi L \quad \text{and} \quad K_L \leq I^{d/4}.
\]

Note that the above conditions holds for \( L = L(I - 1) \) and thereby there exists at least one \( L \) satisfying it. Also note that it holds for all \( I \) if we put \( L = 1 \). Because of \( K_{L(1)} + \cdots + K_{L(I-1)+1} + 1 \geq I \to \infty \) and \( I^{d/4} \to \infty \), for all \( L \) the above inequalities holds for large enough \( I \), and thereby \( L(I) \to \infty \) as \( I \to \infty \). Put \( \Lambda(I) = K_{L(1)} + \cdots + K_{L(I)} \). We have \( \rho_{\Lambda(I-1)+1} \leq \pi L(I) \) and \( K_{L(I)} \leq I^{d/4} \).

Let define a sequence \( \lambda(I) \) of integers inductively by \( \lambda(1) = 1 \) and recursive formula \( \lambda(I + 1) = 1 + \lambda(I) + \lceil \log_2 (1 + \rho_1) m_{K(I)}^{(L(I))} K_{L(I)}^2 I^3 \rceil \).

Let us put \( Y_I(x) = Z_{L(I)}(2^{\lambda(I)}x) \), \( c_{\lambda(I)+j} = c_j^{(L(I))} \), \( n_{\lambda(I)+j} = 2^{\lambda(I)} m_j^{(L(I))} (1 \leq j \leq K_{L(I)}) \), and \( S_N(x) = \sum_{k=1}^N c_k f(n_k x) \). Obviously, \( Y_I = \sum_{k=\lambda(I)+1}^{\Lambda(I)} c_k f(n_k \cdot) = S_{\Lambda(I)} - S_{\lambda(I)-1} \) and \( S_{\lambda(I)} \) let us denote \( Y_1 + \cdots + Y_I \).

Depending on \( \Lambda(I-1) < k < \Lambda(I) \) or \( k = \Lambda(I) \), we can verify; \( n_{k+1}/n_k = m_{k-\Lambda(I-1)+1}^{(L(I))}/m_{k-\Lambda(I)-1}^{(L(I))} \geq 1 + \pi L(I) \geq 1 + \rho L(I-1)+1 \geq 1 + \rho_k \), \( n_{k+1}/n_k = 2^{\lambda(I)+1} m_1^{(L(I))+1}/2^{\lambda(I)} m_1^{(L(I))} \geq (1 + \rho_1) m_1^{(L(I)+1)} K_{L(I)}^2 I^3 \geq 1 + \rho_k \).

We estimate \( \omega_2(\delta; Y_I) \). By applying \( \omega_2(\delta; X(N \cdot)) = \omega_2(N\delta; X) \), we have \( \omega_2(\delta; Z_L) \leq \omega_2(\delta; f(m_1^{(L)} \cdot)) + \cdots + \omega_2(\delta; f(m_{K_L}^{(L)} \cdot)) \leq K_L \omega_2(\delta m^{(L)}_1; f) \) and \( \omega_2(\delta; Y_I) = \omega_2(2^{\lambda(I)}\delta; Z_L(I)) \). By \( f \in \text{Lip}^2 1/2 \), we have \( \omega_2(\delta; Y_I) \leq C K_{L(I)}(2^{\lambda(I)} \omega^{(L(I))} m_1^{(L(I))})^{1/2} \).

From now on, we regard unit interval with Lebesgue measure as a probability space and we use the standard notation of expectation, conditional expectation, and so on.

Let \( d_i(x) \) be an \( i \)-th digit under the decimal point of the binary expansion of \( x \), i.e., \( x = \sum_{i=1}^\infty d_i(x)/2^i \) (mod 1). Let \( \mathcal{F}_n \) be a \( \sigma \)-field on \([0,1]\) generated by \( \{(k-1)/2^n, k/2^n) \mid k = 1, \ldots, 2^n \} \).

Put \( X_I = E(Y_I \mid \mathcal{F}_{\lambda(I+1)-1}) \). Since \( Y_I \) is a periodic function with period \( 2^{-\lambda(I)} \), \( X_I \) have the same period, and is constant over each interval \( [(k-1)2^{-\lambda(I)+1}, k2^{-\lambda(I)+1}) \) \( (k = 1, \ldots, 2^{\lambda(I+1)-1}) \). Thus we can express \( X_I \) as \( X_I = F_I(d_{\lambda(I)+1}, \ldots, d_{\lambda(I+1)-1}) \) where \( F_I \) is a Borel function. Since \( \{d_i\} \) forms an i.i.d., the sequence \( U = \sum_{i=1}^\infty d_{\lambda(I)}/2^I \), \( X_1, X_2, \ldots \) is independent. Note that \( U \) is a uniformly distributed random variable.

By applying \( \|X - E(X \mid \mathcal{F}_n)\|_2 \leq C \omega_2(2^{-n}; X) \), we have

\[
\|Y_I - X_I\|_2 \leq C \omega_2(2^{-\lambda(I+1)}; Y_I) = O(I^{-3/2}),
\]
and hence $E \sum_I |X_I - Y_I| \leq \sum_I E|X_I - Y_I| \leq \sum_I E^{1/2}|X_I - Y_I|^2 < \infty$. Therefore $\sum_I |X_I - Y_I| < \infty$ a.e. and $E^{1/2}\sum_I(X_I - Y_I)^2 < \infty$. Since our $X_1$, $X_2$, ... is independent sequence with $EX_j = 0$, we have

$$V_I := \sum_{j=1}^{I} E(X_j^2 | F_{j-1}) = E \left( \sum_{j=1}^{I} X_j^2 \right) = E \left( \sum_{j=1}^{I} Y_j^2 \right) + O(1) = v_{\Lambda(I)} + O(1).$$

Because of $|EX^2 - EY^2| \leq E^{1/2}|X - Y|^2(2E^{1/2}X^2 + E^{1/2}|X - Y|^2)$ and $\|X_I\|_r \leq \|Y_I\|_r \leq K(L(I))\|f\|_r = O(I^{d/4})$, $(r \geq 1)$, we have the estimate

$$\sum_I |EX^2_I - EY^2_I| = \sum_I O(I^{-3/2}(I^{d/4} + I^{-3/2})) = \sum_I O(I^{-5/4}) = O(1),$$

hence $V_I = \sum_{j=1}^{I} EX^2_j = \sum_{j=1}^{I} EY^2_j + O(1) \geq \|f\|^2_2 \sum_{j=1}^{I} K_L(j)L(j) + O(1).$

Because of $L(I) \rightarrow \infty$, we have $\Lambda(I) = \sum_j^{I} K_L(j) = o(\sum_j^{I} K_L(j)L(j))$ and hence $I \leq \Lambda(I) = o(V_I) = o(v_{\Lambda(I)}), or V_I^{-1} = o(I^{-1})$ and $v_{\Lambda(I)}/\Lambda(I) \rightarrow \infty$. Thus we have $v_{\Lambda(I-1)} - v_{\Lambda(I)} = EX^2_I + O(1) = O(I^{d/2}) = o(v_{\Lambda(I)}), and hence v_{\Lambda(I-1)} \sim v_{\Lambda(I)}$, and consequently, $v_{N-1} \sim v_N$ and $v_N/N \rightarrow \infty$. We use the following result by Monrad-Philipp [13].

**Theorem A.** Let $\{X_I, F_I\}_I$ be a real-valued square-integrable martingale difference sequence. Let $\Phi$ be a non-decreasing function such that $\Phi(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $\Phi(x)(\log x)\delta/x$ is non-increasing for some $\delta > 50$. Let us assume $V_I = \sum_{j=1}^{I} E(X_j^2 | F_{j-1}) \rightarrow \infty$ a.s and $\sum_I E(X_I^2 1_{t \geq v_{\Lambda(I)}})/\Phi(V_I)) < \infty$. Suppose that there exists a uniform random variable which is independent of $\{X_I\}$. Then there exists a standard normal i.i.d. $\{G_I\}$ such that

$$\sum_{j \geq 1} X_j 1_{V_I \leq t} = \sum_{j \leq t} G_j + o(t^{1/2}(\Phi(t)/t)^{1/50}), \quad (t \rightarrow \infty), \quad \text{a.s}$$

We apply the theorem for $\Phi(x) = x^d$. We have already proved $V_I \rightarrow \infty$. Because of $E(X_I^2 1_{t \geq \Phi(V_I)})/\Phi(V_I) \leq E(|X_I|^p/\Phi(V_I)^{p/2}) = O(I^{-dp/4})$, it is summable in $I$, and we can apply Theorem A. By putting $t = V_I$, we have $X_1 + \cdots + X_I = W_{[V_I]} + o(V_I^c)$ as $I \rightarrow \infty$, a.s. By $V_I = v_{\Lambda(I)} + O(1)$, we have $W_{[V_I]} = W_{v_{\Lambda(I)}} + o(v_{\Lambda(I)}^c)$. Thus, by $\sum_{j} Y_j = \sum X_j + O(1)$, we have $Y_1 + \cdots + Y_I = W_{v_{\Lambda(I)}} + o(v_{\Lambda(I)}^c)$ or $S_{\Lambda(I)} = W_{v_{\Lambda(I)}} + o(v_{\Lambda(I)}^c)$ as $I \rightarrow \infty$, a.s. For $M_I = \max_j^{K(I)} |S_{\Lambda(I-1)+j} - S_{\Lambda(I-1)}|$, we have $EM_I^p \leq E \sum_j^{K(I)} |S_{\Lambda(I-1)+j} - S_{\Lambda(I-1)}|^p \leq K(I)(K(I)\|f\|_p)^p = O(I^{d(1+p)/4})$. For $\widetilde{M}_I = \max_{v=v_{\Lambda(I)}}^{v_{\Lambda(I)}} |W_v - W_{v_{\Lambda(I)}}|$, by Levy’s inequality we have $E\widetilde{M}_I^p \leq CE|W_{v_{\Lambda(I)}} - W_{v_{\Lambda(I)-1}}|^p = C'(v_{\Lambda(I)} - v_{\Lambda(I)-1})^{p/2} = O(I^{dp/4}) = O(I^{d(1+p)/4})$. Thus $E(M_I/v_{\Lambda(I)}^c)^p = O(I^{d(1+p)/4-p(1/2-1-d)/50})$, which is summable in $I$ and thereby $\sum_I(M_I/v_{\Lambda(I)}^c)^p < \infty$ a.s. and hence $M_I = o(v_{\Lambda(I)}^c)$ a.s. In the same way we have $\widetilde{M}_I = o(v_{\Lambda(I)}^c)$ a.s. Therefore, we have the desired almost sure invariance principle $S_N = W_{v_N} + o(v_{N}^c)$ as $N \rightarrow \infty$, a.s.
4. Proof of part (2) of Theorem

We prove assuming that $a(\nu) = (-1)^\nu \Re \hat{f}(\nu) \geq 0$, $b(\nu) = (-1)^\nu \Im \hat{f}(\nu) \geq 0$.

Let $p_0 = 2$, $p_1 = 3$, \ldots are sequence of whole prime numbers. We have

$$
\int_0^1 f(nx) f(mx) \, dx = \sum_{\nu=1}^{\infty} \left( \hat{a}(\nu m) \hat{a}(\nu n) + \hat{b}(\nu m) \hat{b}(\nu n) \right) \geq 0 \text{ for odd } n \text{ and } m \text{ with } \gcd(n, m) = 1.
$$

Hence applying the argument section 2 by putting $H = 0$ and $e_k = 1$, or $X_{p,k} = 1$, by $I_{j_1^+ \ldots j_r^+} \geq 0$, we can apply the monotone convergence theorem instead of Lebesgue’s convergence theorem and have $\lim_{J \to \infty} \frac{1}{f \int_0^1 [\sum_{j_1, \ldots, j_r=0}^{J-1} f(p_{j_1} \ldots p_{j_r}^x)]^2 \, dx = \sum_{j_1, \ldots, j_r \in \mathbb{Z}} I_{j_1^+ \ldots j_r^+}^J}$. By using the above expression of $\int_0^1 f(nx) f(mx) \, dx$, and by noting that it is the non-negative series, we can change the order of summation and have

$$
\sum_{j_1, \ldots, j_r \in \mathbb{Z}} I_{j_1^+ \ldots j_r^+}^J = \sum_{c=a,b} \sum_{k=1}^{\infty} \left( \sum_{j_1, \ldots, j_r \in \mathbb{N}_0} \tilde{c}(p_{j_1} \ldots p_{j_r}^x 2^k) \right)^2.
$$

Therefore

$$
\lim_{J \to \infty} \frac{1}{f \int_0^1 [\sum_{j_1, \ldots, j_r=0}^{J-1} f(p_{j_1} \ldots p_{j_r}^x)]^2 \, dx \geq \left( \sum_{j_1, \ldots, j_r \in \mathbb{N}_0} \tilde{a}(p_{j_1} \ldots p_{j_r}^x) \right)^2 \to \infty \text{ as } \tau \to \infty,
$$

because of $\sum_{n \in B_1^r} \tilde{a}(\nu) \uparrow \sum \tilde{a}(\nu) = \infty$. Thus we can apply the argument of previous section and prove our result.

References


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