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The law of the iterated logarithm
for the discrepancies of a permutation of \{n_k x\}

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Abstract. For any unbounded sequence \{n_k\} of positive real numbers, there exists a permutation \{n_{\sigma(k)}\} such that the discrepancies of \{n_{\sigma(k)} x\} obey the law of the iterated logarithm exactly in the same way as the uniform i.i.d. sequence \{U_k\}.

1. Introduction
In the theory of uniform distribution, the following two types of discrepancies of a sequence \{x_k\} of real numbers are frequently used:

\[ D_N(x_k) = \sup_{0 \leq a' < a < 1} \left| \sum_{k=1}^{N} f_{a',a}(x_k) \right|; \quad D^{*}_N(x_k) = \sup_{0 \leq a < 1} \left| \sum_{k=1}^{N} f_{0,a}(x_k) \right|, \]

where \( f_{a',a}(x) = 1_{[a',a)}(\langle x \rangle) - (a-a') \) denotes the indicator function of \([a',a)\) and \( \langle x \rangle \) denotes the fractional part \( x - \lfloor x \rfloor \) of real number \( x \).

We are interested in the asymptotic behavior as \( N \to \infty \) of discrepancies. For uniform i.i.d. \{U_k\}, the law of the iterated logarithm holds (Cf. [4]):

\[ \lim_{N \to \infty} \frac{N D_N(U_k)}{\sqrt{2N \log \log N}} = \frac{1}{2} \text{ a.s.} \]

Assuming the Hadamard’s gap condition \( n_{k+1}/n_k > q > 1 \), Philipp [5, 6] proved the following asymptotic property and solved the Erdős-Gál conjecture:

\[ \frac{1}{4\sqrt{2}} < \lim_{N \to \infty} \frac{N D_N(n_k x)}{\sqrt{2N \log \log N}} \leq C_q \quad \text{a.e.,} \]

where \( C_q \) is a constant depending only on \( q \). For special sequence \( \{2^k\} \), exact law of the iterated logarithm below is proved in [3]:

\[ \lim_{N \to \infty} \frac{N D_N(2^k x)}{\sqrt{2N \log \log N}} = \lim_{N \to \infty} \frac{N D^{*}_N(2^k x)}{\sqrt{2N \log \log N}} = \frac{\sqrt{42}}{9} \quad \text{a.e.} \]

For uniform i.i.d. \{U_k\}, the law of the iterated logarithm for discrepancies holds for every permutation of \{U_k\}.

In a recent literature [2], Berkes, Philipp and Tichy made a remark that Philipp’s asymptotic property above is permutation-invariant under Hadamard’s gap condition, i.e., it remains valid if we permute the order of \( \{n_k\} \). Relating to this remark, we show that the values of \( \limsup \) itself are not permutation-invariant in general.

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Theorem. For any unbounded sequence \( \{n_k\} \) of positive real numbers, there exists a bijective transformation \( \sigma \) on \( \mathbb{N} \) such that

\[
\lim_{N \to \infty} \frac{N D_N \{n_{\sigma(k)} x\}}{\sqrt{2N \log \log N}} = \lim_{N \to \infty} \frac{N D_N^* \{n_{\sigma(k)} x\}}{\sqrt{2N \log \log N}} = \frac{1}{2} \quad \text{a.e.}
\]

For the sequence \( \{2^k\} \) and \( a = 2, 3, \ldots \), there exists a \( \sigma \) with

\[
\lim_{N \to \infty} \frac{N D_N \{2^{\sigma(k)} x\}}{\sqrt{2N \log \log N}} = \lim_{N \to \infty} \frac{N D_N^* \{2^{\sigma(k)} x\}}{\sqrt{2N \log \log N}} = \frac{1}{2} \sqrt{\frac{(2^a + 1)2^a(2^a - 2)}{(2^a - 1)^3}} \quad \text{a.e.}
\]

In this theorem, \( \{n_k\} \) may not be integers nor increasing.

2. LIL for the case of large gap

In this section we prove the proposition below.

Proposition. For any sequence \( \{n_k\} \) of positive real numbers satisfying \( n_{k+1}/n_k \to \infty \), we have

\[
\lim_{N \to \infty} \frac{N D_N \{n_k x\}}{\sqrt{2N \log \log N}} = \lim_{N \to \infty} \frac{N D_N^* \{n_k x\}}{\sqrt{2N \log \log N}} = \frac{1}{2} \quad \text{a.e.}
\]

Proof: By applying the result by Berkes [1], for any \( a' < a \), we have

\[
\lim_{N \to \infty} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^{N} f_{a',a}(n_k x) \right| = \|f_{a',a}\|_2 = \sqrt{(a - a')(1 - (a - a'))}.
\]

Hence we can verify

\[
\lim_{N \to \infty} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^{N} f_{2^{-L}I',2^{-L}I}(n_k x) \right| = \frac{1}{2} \quad \text{a.e.}
\]

Put \( \Psi_{L,I,N}(x) = \sup_{0 \leq a < 2^{-L}} \left| \sum_{k=1}^{N} f_{2^{-L}I',2^{-L}I+a}(n_k x) \right| \). In the same way as the proof in section 3 of [3], which originated to [5], we can prove

\[
\lim_{N \to \infty} \frac{\Psi_{L,I,N}(x)}{\sqrt{2N \log \log N}} \leq C 2^{-L/8} \quad \text{a.e.} \quad (L \in \mathbb{N}, I = 0, \ldots, 2^L - 1).
\]

On the other hand, we can easily verify the approximation inequality below:

\[
\left| \sup_{0 \leq a' < a < 1} \sum_{k=1}^{N} f_{a',a}(n_k x) - 2^{L-1} \max_{I' = 0} \max_{I = 1}^{I'} \sum_{k=1}^{N} f_{2^{-L}I',2^{-L}I}(n_k x) \right| \leq 2 \max_{I = 0}^{2^{L-1}} \Psi_{L,I,N}(x)
\]

By combining these and letting \( L \to \infty \), we have the conclusion. As to \( D_N^* \), we can prove our result in the same way.
3. Proof of the Theorem

Since \( \{n_k\} \) is unbounded, we can take a subsequence \( \{n_{i_k}\} \) such that \( n_{i_{k+1}}/n_{i_k} \geq k \). We make a subsequence \( \{n_{j_k}\} \) by removing \( \{n_{i_k}\} \) from \( \{n_k\} \). Hence we have divided \( \{n_k\} \) into two subsequences \( \{n_{i_k}\} \) and \( \{n_{j_k}\} \).

Let us define \( \sigma \) as follows. Put \( \sigma(j_k) = k(k+1)/2 \). For \( l = 1, 2, \ldots \) and \( k \) satisfying \( (l-1)l/2 < k \leq l(l+1)/2 \), put \( \sigma(i_k) = k + l \), which varies over \( l(l+1)/2 + 1, \ldots , (l+1)(l+2)/2 - 1 \). We can verify that \( \sigma \) is bijective transformation on \( \mathbb{N} \), and that \( b_N := \#\{k \mid \sigma(j_k) \leq N\} = O(\sqrt{N}) \).

By definition of \( \sigma \), we have

\[
\sum_{k=1}^{N} f_{a',a}(n_{\sigma(k)} x) = \sum_{k=1}^{N-b_N} f_{a',a}(n_{i_k} x) + \sum_{k=1}^{b_N} f_{a',a}(n_{j_k} x).
\]

Since the last term is of \( O(b_N) = O(\sqrt{N}) \), by definition of \( D_N \), we have

\[
ND_N\{n_{\sigma(k)} x\} = (N - b_N)D_N\{n_{i_k} x\} + O(\sqrt{N}).
\]

It also holds for \( D_N^* \). Since \( \{n_{i_k}\} \) satisfies the condition of Proposition, we have

\[
\lim_{N \to \infty} \frac{ND_N^*\{n_{i_k} x\}}{\sqrt{2N \log \log N}} = \lim_{N \to \infty} \frac{ND_N\{n_{i_k} x\}}{\sqrt{2N \log \log N}} = \frac{1}{2} \text{ a.e.}
\]

Thanks to \( b_N = O(\sqrt{N}) \), we have \( N - b_N \sim N, \sqrt{N} = o(\sqrt{2N \log \log N}) \), and \( \sqrt{2N \log \log N} \sim \sqrt{2(N - b_N) \log \log (N - b_N)} \), we have the conclusion.

For the binary sequence \( \{2^k\} \), put \( i_k = ak \) and define \( \{j_k\} \) and \( \sigma \) as above. As to the sequence \( \{2^{ak}\} (a = 2, 3, \ldots) \), we have the LIL below (Cf. [3]).

\[
\lim_{N \to \infty} \frac{ND_N\{2^{ak} x\}}{\sqrt{2N \log \log N}} = \lim_{N \to \infty} \frac{ND_N^*\{2^{ak} x\}}{\sqrt{2N \log \log N}} = \frac{1}{2} \sqrt{\frac{(2^a + 1)2^a(2^a - 2)}{(2^a - 1)^3}} \text{ a.e.}
\]

Hence we have the conclusion in the same way.
References


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