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A metric discrepancy result for the sequence of powers of minus two

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Abstract

The law of the iterated logarithm for discrepancies of \( f(2^k t) \) is proved. This result completes the concrete determination of the law of the iterated logarithm for discrepancies of the geometric progression with integer ratio, and reveals the fact that 2 is the only positive integer \( \theta > 1 \) such that fractional parts of \( (-\theta^k t)_k \) converges to uniform distribution faster than those of \( (\theta^k t)_k \) a.e. \( t \).

Keywords: discrepancy, lacunary sequence, law of the iterated logarithm

2000 MSC: 11K38, 42A55, 60F15

1. Introduction

Kronecker [23] proved that the sequence of the fractional part of \( kt \) \((k = 1, 2, \ldots)\) is dense in the unit interval if and only if \( t \) is irrational, and it was more than twenty years later that Bohl [7], Sierpiński [29] and Weyl [32] proved independently that the sequence is uniformly distributed modulo one in the following sense: A sequence \( f(x)_k \) of real numbers is said to be uniformly distributed modulo one if

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{\{a \leq \langle x \rangle_n < b\}} = \frac{b-a}{\log_2(1 + b-a)}
\]

uniformly for all \( a, b \in [0, 1) \), where \( \langle x \rangle \) denotes the fractional part \( x - \lfloor x \rfloor \) of real number \( x \). These results initiated the theory of uniform distribution.

Weyl proved \( D_N f(nk) \to 0 \) a.e. \( t \) under very mild condition \( n_{k+1} - n_k > C > 0 \) for all large \( k \), and showed that the method of measure theory is effective in the research of the uniform distribution theory.

Various studies were done in this direction. For arithmetic progressions \( (kt) \) and increasing functions \( g \), Khintchine [21] proved that

\[
D_N^k f(kt) = O((\log N)g(\log \log N)) \quad \text{a.e. } t
\]
holds if and only if the function $g$ satisfies $\sum 1/g(n) < \infty$. When $\sum 1/g(n) < \infty$ is satisfied, we can easily derive a stronger result
\[
ND_N^*([k]) = o ((\log N)g(\log N)) \quad \text{a.e. } t,
\]
and see that critical speed cannot be determined in almost everywhere sense. The critical speed was determined by Kesten [20] in the sense of convergence in measure:
\[
\lim_{N \to \infty} \text{Leb}\{t \in [0, 1) \mid \frac{ND_N^*([k])}{\log N \log \log N} = \frac{2}{\pi^2} > \varepsilon\} = 0, \quad (\varepsilon > 0).
\]
In probability theory, the following beautiful result was proved by Chung [8] and Smirnov [30] independently, viz. the law of the iterated logarithm
\[
\lim_{N \to \infty} \frac{ND_N^*([k])}{\sqrt{2N \log \log N}} = \lim_{N \to \infty} \frac{ND_N([U_k])}{\sqrt{2N \log \log N}} = \frac{1}{2} \text{ a.s.}
\]
where $\{U_k\}$ is the sequence of independent and uniformly distributed random variables.

After a number of studies on the behaviour of $ND_N^*([k])$ for increasing $[k]$, Erdős [11] conjectured $ND_N^*[n_k] = O((N \log \log N)^{1/2})$ a.e. assuming the Hadamard gap condition $n_{k+1}/n_k \geq q > 1$. Since the law of the iterated logarithm
\[
\lim_{N \to \infty} \frac{1}{\sqrt{2N \log \log N}} \sum_{k=1}^{N} \cos 2\pi n_k t = \frac{1}{\sqrt{2}} \text{ a.e. } t
\]
was proved under the Hadamard gap condition by Erdős-Gál [12], it was natural to expect the analogue of the Chung-Smirnov result above.

By using Takahashi’s method [31], Philipp [26] solved the conjecture by showing the bounded law of the iterated logarithm
\[
\frac{1}{4 \sqrt{2}} \leq \lim_{N \to \infty} \frac{ND_N^*[n_k]}{\sqrt{2N \log \log N}} \leq \lim_{N \to \infty} \frac{ND_N([n_k])}{\sqrt{2N \log \log N}} \leq \frac{1}{\sqrt{2}} \left(166 + \frac{664}{q^{1/2} - 1}\right) \text{ a.e. } t.
\]
For a proof using martingales and another approach, see Philipp [25, 27]. Dhompongsa [9] assumed the very strong gap condition
\[
\log(n_{k+1}/n_k)/ \log \log k \to \infty \quad (k \to \infty)
\]
and derived the Chung-Smirnov type result
\[
\lim_{N \to \infty} \frac{ND_N^*[n_k]}{\sqrt{2N \log \log N}} = \lim_{N \to \infty} \frac{ND_N([n_k])}{\sqrt{2N \log \log N}} = \frac{1}{2} \text{ a.e. } t.
\]
The condition was relaxed later [1, 14] to $n_{k+1}/n_k \to \infty$.

On the other hand, Berkes-Philipp [5] proved that for any $\varepsilon_k \to 0$ there exists $[n_k]$ with $n_{k+1}/n_k \geq 1 + \varepsilon_k$ and
\[
\lim_{N \to \infty} \frac{ND_N^*[n_k]}{\sqrt{2N \log \log N}} = \lim_{N \to \infty} \frac{ND_N([n_k])}{\sqrt{2N \log \log N}} = \infty \text{ a.e. } t.
But there still exist sequences obeying the bounded law of the iterated logarithm which do not satisfy the Hadamard gap condition. Indeed, Philipp [28] proved that the multiplicative semigroups generated by finitely many coprime integers (Hardy-Littlewood-Pólya sequences) are such examples. For permutational law of the iterated logarithm for Hardy-Littlewood-Pólya sequences, we refer the reader to Aistleitner-Berkes-Tichy [2]. See also [6, 19]. For other permutational law of the iterated logarithm results, see Aistleitner-Berkes-Tichy [3].

The studies for geometric progressions \( \{\theta t\} \) were not in the same detail as those for arithmetic progressions. But recently, many concrete values of the limsups were evaluated [13, 15, 17]. For any \( \theta \not\in [-1, 1] \), there exist real numbers \( \Sigma_0^+ \) and \( \Sigma_0^- \) such that

\[
\lim_{N \to \infty} \frac{ND_N[3\theta t]}{\sqrt{2N \log \log N}} = \Sigma_0^+ \quad \text{and} \quad \lim_{N \to \infty} \frac{ND_N[3\theta t]}{\sqrt{2N \log \log N}} = \Sigma_0^- \quad \text{a.e. } t.
\]

When \( \theta \not\in [-1, 1] \) is not a root of a rational number, we have \( \Sigma_0^+ = \Sigma_0^- = \frac{1}{2} \), while we have \( \Sigma_0^+ = \Sigma_0^- > \frac{1}{2} \) when \( \theta > 1 \) is a root of a rational number.

We give the concrete expressions in case when \( \theta \) is an integer. The constant \( \Sigma_0^- \) is equal to \( \frac{1}{2} \sqrt{\frac{\theta+1}{\theta-1}} \) when \( \theta > 1 \) is an odd integer, is equal to \( \frac{1}{2} \sqrt{\frac{\theta+1}{\theta-1}} \) when \( \theta \geq 4 \) is an even integer, and is equal to \( \frac{\sqrt{2}}{2} \) when \( \theta = 2 \). When \( \theta \leq -3 \) is an integer, then \( \Sigma_0^- = \Sigma_0^- \). When \( \theta < -1 \) is an odd integer then \( \Sigma_0^- = \frac{1}{2} \sqrt{\frac{\theta+1}{\theta-1}} \), while \( \Sigma_0^+ = \frac{1}{2} \) when \( \theta < -1 \) is even. Among these, the concrete value of \( \Sigma_{-2} \) is missing.

We deal with the case \( \theta = -2 \) in this paper. In [17], we already proved the law of the iterated logarithm as follows. When \( r \) is even,

\[
\lim_{N \to \infty} \frac{ND_N[3(-\sqrt{2} t)]}{\sqrt{2N \log \log N}} = \lim_{N \to \infty} \frac{ND_N[3(-\sqrt{2} t)]}{\sqrt{2N \log \log N}} = \Sigma_2^+ = \sqrt{\frac{42}{9}} \quad \text{a.e. } t,
\]

while, when \( r \) is odd

\[
\lim_{N \to \infty} \frac{ND_N[3(-\sqrt{2} t)]}{\sqrt{2N \log \log N}} = \frac{1}{2} \lim_{N \to \infty} \frac{ND_N[3(-\sqrt{2} t)]}{\sqrt{2N \log \log N}} = \Sigma_2^- \quad \text{a.e. } t. \tag{1.1}
\]

We shall compute \( \Sigma_{-2} \).

**Theorem 1.** For odd \( r \), we have

\[
\lim_{N \to \infty} \frac{ND_N[3(-\sqrt{2} t)]}{\sqrt{2N \log \log N}} = \sqrt{\frac{910}{49}} \quad \text{a.e. } t. \tag{1.2}
\]

We can conclude that 2 is the only positive integer \( \theta \) such that \( \Sigma_0^+ \neq \Sigma_0^- \). Our evaluation \( \Sigma_{-2}/\Sigma_2 = 0.85 \ldots \) proves that the distribution of the fractional parts of \( (-2)^kt \) tends to the uniform distribution about 15% faster than that of \( [2^kt] \) for a.e. \( t \).

Before closing the introduction, we call attention to our recent result [4] proving the exact law of the iterated logarithm for the discrepancies of sequences \( [n_t] \) satisfying the Hadamard gap condition \( n_{q+1}/n_q \geq q + 1 \) and a very mild Diophantine condition. The constants appearing there are bounded from above by \( \frac{1}{2} \sqrt{\frac{q+1}{q-1}} \), which is identical with \( \Sigma_0 \) in case \( q \) is an odd integer. Without any Diophantine condition, we [18] were able to derive the slightly loose upperbound \( \frac{1}{2} \sqrt{\frac{1}{N^{6/5}}} \). For other relating results, see [16].
2. Preliminaries

First we recall the expression

\[ \Sigma_2 = \left( \sup_{0 \leq x < 1} v(x, y) \right)^{1/2}, \]  

(2.1)

which is proved in [17]. Here \( v(x, y) \) is a continuous function on \( \mathbb{R}^2 \) defined as the following absolutely and uniformly convergent series.

\[ \tilde{V}(x, y) = v(x, y) + 2 \sum_{k=1}^{\infty} \left( \tilde{V}(\frac{-2^{k-1}x}{2^k}, \frac{2^k y}{2^k}, (x), (y)) \right). \]

where the function \( \tilde{V} \) is defined as

\[ \tilde{V}(\xi, x) = \tilde{V}(\xi, x, y) = v(\xi, x) \]  

(2.2)

for \( x, y, \xi, \eta \in [0, 1) \). Here we write \( a \lor b = \max\{a, b\} \) and \( a \land b = \min\{a, b\} \). Clearly we have \( \tilde{V}(\xi, x) = v(\xi, x) \) and

\[ \tilde{V}(\xi, \eta, x, y) = -\tilde{V}(\xi, \eta, x, y) = -\tilde{V}(\eta, x, y) \]

(2.4)

Therefore we can express \( v(x, y) \) simply by

\[ v(x, y) = \tilde{V}((x), (y), (x), (y)) + 2 \sum_{k=0}^{\infty} \tilde{V}\left(\left(\frac{-2^k x}{2^{k+1}}, \frac{2^k y}{2^{k+1}}\right) \right). \]

(2.5)

We prove the following formulas for \( \xi, \eta, x, y \in \mathbb{R} \) and \( c \in \mathbb{R} \):

\[ \tilde{V}(\xi+c, \eta+c, (x+c), (y+c)) = \tilde{V}(\xi, \eta, (x), (y)), \]

(2.6)

\[ \tilde{V}(\eta-x, \eta, (y-x), (y-x)) = \tilde{V}(\xi, \eta, (x), (y)), \]

(2.7)

\[ \tilde{V}(\xi, \eta, (x), (y)) = \tilde{V}(\eta-x, (y-x), (y-x)), \]

(2.8)

\[ \tilde{V}(\xi, x, y) = \tilde{V}(\xi, x, y) \]

(2.9)

\[ V(\xi, x) = \tilde{V}(\xi, x) = \tilde{V}(\xi, x) \]

(2.10)

where \( a^+ \) denotes \( a \land 0 \). First, for \( x, y, t \in \mathbb{R} \) satisfying \( 0 \leq x - y < 1 \), we put \( I_{x,y}(t) = \sum_{n \in \mathbb{Z}} I_{(x,y)}(t+n) \) and \( I_{\eta,\xi}(t) = I_{\eta,\xi}(t) - (y-x) \), where \( I_{(x,y)} \) is the indicator function of \( [x, y) \). If \( 0 \leq x < y < 1 \), we see \( I_{x,y}(t) = I_{(x,y)}(t) \). We have

\[ I_{x,y}(t) = \tilde{I}_{x,y}(t) - \tilde{I}_{x,y}(t) \quad (y-x \in [0,1]). \]

(2.11)

Actually, it is almost trivial when \( \langle x \rangle \leq \langle y \rangle \), and otherwise it is verified by \( I_{x,y} = 1 - I_{\langle y \rangle,\langle x \rangle} \) and \( y-x = (y) - \langle y \rangle + 1 \). We can show

\[ \int_0^1 \tilde{I}_{\eta,\xi}(t) \tilde{I}_{x,y}(t) \, dt = \tilde{V}(\xi, \langle y \rangle, (x), (y)) \quad (\eta - \xi - y - x \in [0,1]). \]

(2.12)
Indeed, \( \int_0^1 I_{0,\xi}(t)I_{0,\xi}(t)\,dt = \langle \xi \rangle \wedge \langle x \rangle \) implies \( \int_0^1 I_{0,\xi}(t)I_{0,\xi}(t)\,dt = V(\xi, \langle x \rangle) \), and hence (2.11) yields (2.12).

We have \( \int_0^1 I_{\xi\eta}(t-c)\tilde{I}_{\xi\eta}(t-c)\,dt \equiv \int_0^1 I_{\xi\eta}(t)\tilde{I}_{\xi\eta}(t)\,dt \), since the integrand has period 1. By noting \( \tilde{I}_{\xi\eta}(t-c) \equiv \tilde{I}_{\xi\eta+c,\eta}(t) \) and \( \tilde{I}_{\xi\eta}(t-c) \equiv \tilde{I}_{\xi\eta+c,\eta}(t) \), we can verify (2.6) assuming \( \eta - \xi, y - x \in [0, 1] \).

For general \( \xi, \eta, x, y \in \mathbb{R} \), we can take \( \tilde{\xi}, \tilde{\eta}, \tilde{x}, \tilde{y} \in \mathbb{R} \) such that \( 0 < \tilde{\xi} < \xi, 0 < \tilde{\eta} < \eta \), and \( x - \tilde{x}, \tilde{x} - \tilde{\xi}, y - \tilde{y} - \tilde{\eta} \in \mathbb{Z} \). By \( \tilde{V}(\langle \xi + c, \eta \rangle, (x+c), (y+c)) = \tilde{V}(\langle \xi, \eta \rangle, (x+c), (y+c)) \) and \( \tilde{V}(\langle \xi, \eta \rangle, (x, y)) = \tilde{V}(\langle \xi, \eta \rangle, (x, y)) \), we see that (2.6) holds for any \( \xi, \eta, x, y \in \mathbb{R} \).

Since we have \( \tilde{I}_{\xi\eta}(t) = \tilde{I}_{\xi\eta-c}(-t) \) and \( \tilde{I}_{\xi\eta}(t) = \tilde{I}_{\xi\eta-c}(-t) \) for almost every \( t \), we have \( \int_0^1 I_{\xi\eta}(t)\tilde{I}_{\xi\eta}(t)\,dt = \int_0^1 \tilde{I}_{\xi\eta}(t)\tilde{I}_{\xi\eta}(t)\,dt \). By changing the variable \( t \) by \(-t\), and by noting (2.12) we have (2.7) for \( \eta - \xi, y - x \in [0, 1] \). We can verify (2.7) for general \( \xi, \eta, x, y \in \mathbb{R} \) in the same way as above. By applying (2.6), we have \( \tilde{V}(\langle \xi, \eta \rangle, (x, y), (\eta - x), 0, (y-x)) \) and (2.8). Clearly, (2.8) implies (2.9). The convenient expression (2.10) is proved by \( \xi - x - (\xi - x)^{+} = (\xi - x) \wedge 0 = \xi \wedge x - x \).

By noting \( \langle (\langle -2 \rangle f(x + \frac{1}{2}) \rangle) = \langle (\langle -2 \rangle f(x + \frac{1}{2}) \rangle) = \langle (\langle -2 \rangle f(x + \frac{1}{2}) \rangle) = \langle (\langle -2 \rangle f(x + \frac{1}{2}) \rangle) \), and by applying (2.6), we have \( \tilde{V}(\langle (\langle -2 \rangle f(x + \frac{1}{2}) \rangle), (\langle -2 \rangle f(x + \frac{1}{2}) \rangle), (x + \frac{1}{2}), (y + \frac{1}{2}) \rangle = \tilde{V}(\langle (\langle -2 \rangle f(x), (\langle -2 \rangle f(x), (x), (y)) \rangle)

It proves
\[
\nu(x + \frac{1}{2}, y + \frac{1}{2}) = \nu(x, y).
\]

By \( (x + 1) = \langle x \rangle, (\langle -2 \rangle f(x + 1)) = \langle (\langle -2 \rangle f(x) \rangle) \), etc., and by (2.4), (2.7), we have
\[
\nu(x + 1, y) = \nu(x, y + 1) = \nu(1 - x, 1 - y) = \nu(1 - y, 1 - x) = \nu(y, x) = \nu(x, y).
\]

Put
\[
\Delta = \{(x, y) \mid y \geq x, x + y \leq 1, 2x + y \geq 1\} \quad \text{and} \quad \Delta^{\#} = \{(x, y) \mid y \geq x, x + y \leq 2, 2x + y \geq 1\}.
\]

Clearly, \( \Delta \subset \Delta^{\#} \subset [0, 1]^2 \). We here prove
\[
\sup_{0 \leq x \leq 1} \nu(x, y) = \sup_{0 \leq x \leq 1} \nu(x, y) = \sup_{(x,y) \in \Delta^a} \nu(x,y) = \sup_{(x,y) \in \Delta} \nu(x,y).
\]

Because of \( \nu(x, y) = 0 \) for all \( 0 \leq x < 1 \), the first equality is trivial. By (2.14) we see \( \sup_{0 \leq x \leq 1} \nu(x, y) = \sup_{(x,y) \in \Delta^a} \nu(x,y) \), and \( \Delta^{\#} = \{(x + 1, y + 1) \mid 0 \leq x, 0 \leq y \leq 1 \} \) and \( \Delta^{\#} = \{(x, y) \mid y \geq x, x + y \leq 1\} \). By (2.14), we see \( \sup_{(x,y) \in \Delta^a} \nu(x,y) = \nu(x, y) \). Note that \( \Delta = \{(x, y) \in \Delta^a \mid y \geq x, x + y \leq 1\} \) and \( \Delta^{\#} = \{(x, y) \in \Delta^a \mid y \geq x\} \). By (2.14), we see that \( \sup_{(x,y) \in \Delta^a} \nu(x,y) = \sup_{(x,y) \in \Delta^a} \nu(x,y) = \sup_{(x,y) \in \Delta^a} \nu(x,y) \), and hence we completed the proof of (2.15).

Note that we have \( x \leq \frac{1}{2}, y \geq \frac{1}{2}, 1 \leq 2y + x \leq 2, 1 \leq 2x + y \leq 2 \) for \( (x,y) \in \Delta, \) and \( x \leq \frac{1}{2}, y \geq \frac{1}{2} \).
\[ y \geq \frac{1}{2}, \quad 1 \leq 2y + x \leq 2, \quad 1 \leq 2x + y \leq 2 \text{ for } (x, y) \in \Delta^8. \] 
(See Fig. 1.) Put

\[ F(\xi, \eta, x, y) = \bar{V}(\xi, \eta, x, y) - \frac{1}{2} \bar{V}((-2\xi, -2\eta, x, y)) + \frac{1}{4} \bar{V}((4\xi, 4\eta, x, y)), \quad (2.16) \]

\[ \Psi(z) = \begin{cases} 
-12z^2 + 7z/4 & \text{if } z \leq 1/3, \\
-12z^2 + 10z - 1/4 & \text{if } 1/3 < z \leq 1/2, \\
-12z^2 + 14z - 3/4 & \text{if } 1/2 < z \leq 2/3, \\
-12z^2 + 17z - 5/4 & \text{if } 2/3 \leq z,
\end{cases} \quad (2.17) \]

\[ \Phi(x, y) = \Psi(|(y) - (x)|). \quad (2.18) \]

By (2.4), we have

\[ F(\xi, \eta, x, y) = -F(\eta, \xi, x, y) = -F(\xi, \eta, y, x). \quad (2.19) \]

It is clear that

\[ F(\xi + i, \eta, x + k, y + l) = F(\xi, \eta, x, y) \quad \text{and} \quad \Phi(x + k, y + l) = \Phi(x, y) \quad (2.20) \]

hold for any integers \( i, j, k, \) and \( l \). We can easily have invariance relations:

\[ F(\xi + 1, \eta, x, y) = F(\xi, \eta, x + 1, y) = F(\eta, \xi, x, y) = F(\xi, \eta, x, y) \quad (2.21) \]

We can also easily verify \( \Psi(1 - z) = \Psi(z) \) and \( |(1 - x) - (1 - y)| = |(y) - (x)| \), and see that \( |(y + 1/2) - (x + 1/2)| \) equals to either \(|(y) - (x)| \) or \( 1 - |(y) - (x)| \). By combining these we have the invariance relations for \( \Phi \):

\[ \Phi(1 - y, 1 - x) = \Phi(y, x) = \Phi(x + 1/2, y + 1/2) = \Phi(x, y). \quad (2.22) \]

By \( 3x = -2(y - x) + 2y + x = -(y - x) + y + 2x \), we have

\[ -(y - x) + 1 \leq 3x \leq -2(y - x) + 2 \quad ((x, y) \in \Delta^8), \quad (2.23) \]

\[ (1 - 3x)^+ \leq y - x \quad ((x, y) \in \Delta^8). \quad (2.24) \]

Actually, (2.24) is clear from (2.23) by noting \( 1 - 3x \leq y - x \) and \( 3y - 2 = 3(y - x) + 3x - 2 \leq y - x \).

For \( z \in [0, 1] \), we can verify

\[ \Psi(z) = \begin{cases} 
-12z^2 + 7z & \text{if } 0 \leq z \leq 1/3, \\
-12z^2 + 10z - 1 & \text{if } 1/3 < z \leq 1/2, \\
-12z^2 + 14z - 3 & \text{if } 1/2 < z \leq 2/3, \\
-12z^2 + 17z - 5 & \text{if } 2/3 \leq z,
\end{cases} \quad (2.25) \]

The next inequality is easily verified.

\[ -z^2 \leq -\frac{10}{7} z + \frac{25}{49}. \quad (2.26) \]

We here state one of the key inequalities: For any \( x, y, \xi, \eta \in \mathbb{R} \), it holds that

\[ F(\xi, \eta, x, y) \leq \Phi(x, y). \quad (2.27) \]
Preparation for the proof of key inequality (2.27). It is enough to prove (2.27) for every $\xi$, $\eta \in \mathbb{R}^2$ and for every $(x, y) \in \Delta$ in view of the invariance relations (2.20), (2.21) and (2.22). For $(x, y) \in \Delta$, we have
\[
\frac{\partial}{\partial \xi} V((\xi, \eta), (x, y)) = \frac{\partial}{\partial \xi} V((\xi, \eta), (x, y)) = 1_{(0, y)}(\xi) - I_{(0, y)}(\xi) + y
\]
and thereby we can verify
\[
\frac{\partial}{\partial \xi} \frac{-1}{2} V((-2\xi), (-2\eta), (x, y)) = -I_{(x, y)}((-2\xi)) + y - x,
\]
and thereby we can verify $\frac{\partial}{\partial \eta} (\xi, \eta, x, y) = (y - x) - I_{(x, y)}((\xi)) - I_{(x, y)}((-2\xi)) - I_{(x, y)}((4\xi))$. Hence $\frac{\partial}{\partial \eta}$ decreases in $\xi$ from positive to negative only if $(\xi) = x$, $(y-2\xi) = x$, or $(4\xi) = x$. The last conditions hold if and only if $(\xi) = \xi_i (i = 1, \ldots, 7)$ where
\[
\xi_1 = x, \xi_2 = \frac{1 - y}{2}, \xi_3 = \frac{2 - y}{2}, \xi_4 = \frac{x}{4}, \xi_5 = \frac{1 + x}{4}, \xi_6 = \frac{2 + x}{4}, \xi_7 = \frac{3 + x}{4}.
\]
And $\frac{\partial}{\partial \eta}$ decreases in $\eta$ from positive to negative only if $(\eta) = \eta_i (i = 1, \ldots, 7)$ where
\[
\eta_1 = y, \eta_2 = \frac{1 - x}{2}, \eta_3 = \frac{2 - x}{2}, \eta_4 = \frac{y}{4}, \eta_5 = \frac{1 + y}{4}, \eta_6 = \frac{2 + y}{4}, \eta_7 = \frac{3 + y}{4}.
\]
For given $(x, y) \in \Delta$, $F(\xi_i, \eta_j, x, y)$ $(i = 1, \ldots, 7)$ are the candidates of maximum of $F(\xi, \eta, x, y)$ in $(\xi, \eta)$. Hence it is enough to prove
\[
F(\xi_i, \eta_j, x, y) \leq \Phi(x, y) \quad ((x, y) \in \Delta^A)
\]
for $i, j = 1, \ldots, 7$. (Although it is enough to prove in $\Delta$, we prove in $\Delta^A$ to make the proof short.) We call it the inequality of type $ij$. We denote $F(\xi_i, \eta_j, x, y)$ simply by $F_{ij}$.

Hierarchy of the system of inequalities. Denote
\[
\psi(t) = V((t - x), (y - x)) \quad \text{and} \quad \phi(t) = 4\psi(t) - 2\psi(-2t) + \psi(4t).
\]
By (2.8), we have
\[
4F(\xi, \eta, x, y) = -\phi(\xi) + \phi(\eta).
\]
Put $\xi_0 = -2y$, $\xi_1 = 4x$, $\eta_0 = -2x$, and $\eta_1 = 4y$. For simplicity, we denote $y - x$ by $z$. We have
\[
\psi(4\xi_0) = \psi(4\xi_1) = \psi(4\xi_2) = \psi(-2\eta_1) = \psi(-2\eta_2) = \psi(\xi_1),
\]
\[
\psi(4\eta_0) = \psi(4\eta_1) = \psi(-2\eta_2) = \psi(-2\eta_3) = \psi(\eta_1),
\]
\[
\psi(-2\xi_0) = \psi(-2\xi_1) = \psi(\eta_2), \quad \psi(-2\eta_0) = \psi(-2\eta_1) = \psi(\xi_2),
\]
\[
\psi(4\xi_1) = \psi(4\xi_2) = \psi(-2\eta_1) = \psi(\xi_0), \quad \psi(4\eta_2) = \psi(4\eta_3) = \psi(-2\xi_1) = \psi(\eta_0),
\]
\[
\psi(4\xi_0) = \psi(\xi_0), \quad \psi(4\eta_1) = \psi(\eta_1).
\]
We give concrete expressions of $\psi(\xi_0)$ and $\psi(\eta_1)$. By $1 \leq 2y + x \leq 2$, we have $(-2y - x) = 2 - 2y - x$, and by (2.10) we have $\psi(\xi_0) = V(2 - 2y - x, z) = z(2y + x - 1) - (3y - 2)^z$. 

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These give the the following expressions in $\Delta^g$:

$$\varphi(\xi_n) = (1 - 3x)z + (3x - 1)^* - (4x - y - 1)^* - (y - 4x)^*,$$

$$\varphi(\eta_n) = (3 + x - 4y)z - (2 - 3y)^* + (x - 4y + 2)^* + (4y - x - 3)^*.$$
Similarly, we have

\[
\begin{align*}
\psi(\xi_0) &= (2y + x - 1)z - (3y - 2)^*, \\
\psi(\xi_1) &= 0, \\
\psi(\xi_2) &= \frac{1}{2}(2x + y - 1)z, \\
\psi(\xi_3) &= \frac{1}{2}(2x + y)z - \frac{1}{2}(3y - 2)^*, \\
\psi(\xi_4) &= \frac{3}{4}xz, \\
\psi(\xi_5) &= \frac{3x - 1}{4}z + \frac{1}{4}(1 - 3x)^*, \\
\psi(\xi_6) &= \frac{3x + 2}{4}z - \frac{1}{4}(4y - x - 2)^*, \\
\psi(\xi_7) &= \frac{3x + 1}{4}z - \frac{1}{4}(4y - x - 3)^*, \\
\psi(\eta_0) &= (3x - 1)z + (1 - 3x)^+, \\
\psi(\eta_1) &= -z^2 + z, \\
\psi(\eta_2) &= \frac{1}{2}(3x - 1)z + \frac{1}{2}(1 - 3x)^*, \\
\psi(\eta_3) &= \frac{3}{2}xz, \\
\psi(\eta_4) &= \frac{1}{4}(4x - y)z + \frac{1}{4}(y - 4x)^*, \\
\psi(\eta_5) &= \frac{1}{4}(4x - y - 1)z + \frac{1}{4}(y - 4x + 1)^+, \\
\psi(\eta_6) &= \frac{1}{4}(4x - y + 2)z - \frac{1}{4}(3y - 2)^*, \\
\psi(\eta_7) &= \frac{1}{4}(4x - y + 1)z.
\end{align*}
\]

We have the inequalities

\[
\begin{align*}
\psi(\eta_2) &\leq \psi(\eta_3), & \psi(\eta_4), \psi(\eta_5) &\leq \psi(\eta_7) \leq \psi(\eta_6), \\
\psi(\xi_2) &\leq \psi(\xi_3), & \psi(\xi_5) &\leq \psi(\xi_1) \leq \psi(\xi_0), \psi(\xi_7).
\end{align*}
\]

Actually, (2.24) implies \(\psi(\eta_2) \leq \psi(\eta_3)\), \(\psi(\xi_2) \leq \psi(\xi_3)\), \(\psi(\xi_2) \leq \psi(\xi_3)\), and \(\psi(\eta_7) \leq \psi(\eta_6)\). By \(y - 4x \leq 4y - 4x\), we have \(\frac{1}{4}(y - 4x)^* \leq z\) and \(\psi(\eta_4) \leq \psi(\eta_7)\). By (2.23), we have \(y - 4x + 1 = z - 3x + 1 \leq 2z\) and \((y - 4x + 1)^* \leq 2z\), and hence we have \(\psi(\eta_7) \leq \psi(\eta_7)\). By \(y - x - 2 = 2z + x + 2y - 2 \leq 2z\), we have \((y - x - 2)^* \leq 2z\) and \(\psi(\eta_4) \leq \psi(\xi_6)\). By \(4y - x - 3 = z + 3(y - 1) \leq z\), we have \((4y - x - 3)^* \leq z\) and \(\psi(\xi_6) \leq \psi(\xi_7)\).

By applying the inequalities above and relations (2.30), we can prove

\[
\begin{align*}
\phi(\alpha_2) &\leq \phi(\alpha_3), & \phi(\alpha_3) &\leq \phi(\alpha_7), & \phi(\alpha_4) &\leq \phi(\alpha_6), & (\alpha = \xi, \eta).
\end{align*}
\]

Actually, by \(\phi(\xi_2) = 4\psi(\xi_2) - 2\psi(\eta_1) + \psi(\xi_3)\) and \(\phi(\xi_1) = 4\psi(\xi_1) - 2\psi(\eta_1) + \psi(\xi_0)\), we have \(\phi(\xi_2) \leq \phi(\xi_3)\). By \(\phi(\xi_2) = 4\psi(\xi_2) - 2\psi(\eta_1) + \psi(\xi_1)\) and \(\phi(\xi_1) = 4\psi(\xi_1) - 2\psi(\eta_1) + \psi(\xi_1)\), we have \(\phi(\xi_1) \leq \phi(\xi_7)\), and other inequalities can be proved in the same way. Moreover, \(4\psi(\eta_0) - 2\psi(\xi_1) = 2(1 - z)z = 4\psi(\eta_7) - 2\psi(\xi_2)\) implies \(\phi(\eta_0) = \phi(\eta_7)\), and \(4\psi(\xi_4) - 2\psi(\eta_1) = 0 = 4\psi(\xi_5) - 2\psi(\eta_2)\) implies \(\phi(\xi_4) = \phi(\xi_5)\). Hence we have

\[
\begin{align*}
\phi(\xi_4) &\leq \phi(\xi_5), & \phi(\xi_0), \phi(\xi_7), &\phi(\eta_0) \leq \phi(\eta_7).
\end{align*}
\]

By noting (2.29) and the inequalities above, we have

\[
F(\xi_{1i}, \eta_{ij}, x, y) \leq F(\xi_{1i}, \eta_{ij}, x, y) \quad (i \in \Xi_1, j \in H_1, k \in \{1, 2, 4\}, l \in \{1, 3, 7\}),
\]

where \(\Xi_1 = H_1 = \{1\}, \Xi_2 = H_3 = \{2, 3\},\) and \(\Xi_4 = H_2 = \{4, 5, 6, 7\}\). It means that it is sufficient to prove the inequalities of types 11, 13, 17, 21, 23, 27, 41, 43, and 47.

Recall that \(\Delta^a\) is invariant under the transformation \((x, y) \mapsto (1 - y, 1 - x)\). By noting (2.21) and denoting \((X, Y) = (1 - y, 1 - x)\), we have \(F(x, \frac{1}{x^2}, x, y) = F(1 - \frac{1}{x^2}, 1 - x, 1 - y, 1 - x) = 9\)
$F(\frac{1+y}{x}, Y, X, Y)$. Thus, if we prove the inequality of type 13, it can be transformed to the inequality of type 21. In the same way, by $F(x, \frac{1+y}{x}, y) = F(1 - \frac{3x^2}{2}, 1 - x, 1, 1) = F(\frac{1}{2}, Y, X, Y)$ and $F(\frac{1+y}{x}, \frac{1+y}{x}, x, y) = F(1 - \frac{3x^2}{2}, 1 - x, 1, 1, 1) = F(\frac{1}{2}, \frac{1}{2}, Y, X, Y)$, the inequalities of types 17 and 27 are transformed to the inequalities of types 41 and 43.

Hence it is sufficient to prove the inequalities of types 11, 13, 17, 23, 27, and 47.

The inequality of type 11. First, we note that it is sufficient to prove $F(x, y, y, y) \leq \Phi(x, y)$ for $(x, y) \in \Delta$. Actually, if we prove this, by (2.21) and (2.22), we have $F(1 - y, 1 - x, 1 - y, 1 - x) \leq \Phi(1 - y, 1 - x) = \Phi(1 - x, 1 - y)$ for $(x, y) \in \Delta$. Since $[(1 - y, 1 - x) \mid (x, y) \in \Delta]$ covers $\Delta^8 \setminus \Delta$, we have $F(x, y, y, y) \leq \Phi(x, y)$ for $(x, y) \in \Delta^8$.

By noting (2.8) and calculating $\psi(\eta_0) - \psi(\xi_0)$ on $\Delta$, we can verify

$$\tilde{V}((-2x, -2y, (x, y)) = \psi(\eta_0) - \psi(\xi_0) = -2z^2 + (1 - 3x)^2 + (3y - 2)^2 \tag{2.31}$$

$\Delta$ is divided into eight pieces $A_1, \ldots, A_8$ by lines $x = \frac{1}{3}$, $y = \frac{1}{3}$, $x - 4y = -2$, $x - 4y = -3$, $4x - y = 0$, and $4x - y = 1$ (Cf. Figure 1). By calculating $\psi(\eta_0) - \psi(\xi_0)$, we have

$$\tilde{V}(4x, 4y, x, y) = \psi(\eta_0) - \psi(\xi_0) \tag{2.32}$$

where

- $-4x^2 + 7x - 3$ on $A_1 = \{(x, y) \mid x < \frac{1}{2}, 3 \leq 4y - x, 0 \leq y - 4x - \frac{3}{2} \leq y\}$,
- $-4x^2 + 6x + 3y$ on $A_2 = \{(x, y) \mid x < \frac{1}{2}, 2 \leq 4y - x < 3, 0 \leq y - 4x - \frac{3}{2} \leq y\}$,
- $-4x^2 + 2z$ on $A_3 = \{(x, y) \mid x < \frac{1}{2}, 2 \leq 4y - x < 3, -1 \leq y - 4x \leq 0, \frac{3}{2} \leq y\}$,
- $-4x^2 - 2x + 5y - 2$ on $A_4 = \{(x, y) \mid x < \frac{1}{2}, 2 \leq 4y - x < 3, -1 \leq y - 4x \leq 0, y \leq \frac{3}{2}\}$,
- $-4x^2 + z$ on $A_5 = \{(x, y) \mid x < \frac{1}{2}, 1 \leq 4y - x < 2, -1 \leq y - 4x \leq 0, y \leq \frac{3}{2}\}$,
- $-4x^2 + 5z - 1$ on $A_6 = \{(x, y) \mid \frac{1}{2} \leq x < 2, 4y - x < 3, -1 \leq y - 4x \leq 0, y \leq \frac{3}{2}\}$,
- $-4x^2 - 4x + y + 1$ on $A_7 = \{(x, y) \mid \frac{1}{2} \leq x < 1, 4y - x < 2, -1 \leq y - 4x \leq 0, y \leq \frac{3}{2}\}$,
- $-4x^2$ on $A_8 = \{(x, y) \mid \frac{1}{2} \leq x < 1, 4y - x < 2, y - 4x - 1 \leq y \leq \frac{3}{2}\}$.

Hence by (2.9), (2.31), (2.32), and (2.25), we can verify the inequality of type 11 as below.

On $A_1$, $4F_{11} = -12z^2 + 17z - 5 \leq 4\Phi(z) = 4\Phi(x, y)$. On $A_2$, $4F_{11} = -12z^2 + 14z + 3 - (1 - 2x - y) \leq -12z^2 + 14z - 3 \leq 4\Phi(z)$. On $A_3$, $4F_{11} = -12z^2 + 12z - 2 \leq 4\Phi(\frac{3}{2}, \frac{1}{2}, Y, X, Y)$. On $A_4$, $4F_{11} = -12z^2 + 10z - 1 + (1 - 2x - y) \leq 12z^2 + 10z - 1 \leq 4\Phi(z)$. On $A_5$, $4F_{11} = -12z^2 + 7z + 2(1 - 2x - y) \leq -12z^2 + 7z \leq 4\Phi(z)$. On $A_6$, by $z \leq \frac{1}{2}$, we have $4F_{11} = -12z^2 + 7z + 2z - 1 < -12z^2 + 7z \leq 4\Phi(z)$. On $A_7$, $4F_{11} = -12z^2 + 7 + (1 - 2x - y) \leq -12z^2 + 7 + 1 \leq 4\Phi(z)$. On $A_8$, $4F_{11} = -12z^2 + 7z - 3z \leq -12z^2 + 7z \leq 4\Phi(z)$.

The inequality of type 13. By $4F_{13} = 4\psi(\eta_1) - 6\psi(\xi_1) + 3\psi(\eta_0) - \psi(\xi_0)$, we have

$$4F_{13} = (18x - 4)z + 1 - 3x + 2(1 - 3x)^2 + (-y + 4x - 1)^2 + (y - 4x)^2,$$

where we used $-(1 - 3x)^2 + (3x - 1)^2 = 3x - 1$. We divide $\Delta^8$ into a few pieces.

When $y - 4x = 0$, we have $x = \frac{1}{3}$, and hence $4F_{13} = 3x(6z - 4) - 3z$. If $z \geq \frac{1}{2}$, by (2.23) we have $4F_{13} \leq (-2z^2 + 2)(6z - 4) - 3z + 3 = -12z^2 + 17z + 5 \leq 4\Phi(z)$, and otherwise we have $4F_{13} \leq (-z + 1)(6z - 4) - 3z + 3 = -6z^2 + 7z - 1 \leq 4\Phi(z)$.

If $y - 4x \leq 0$ and $x \leq \frac{1}{2}$, we have $4F_{13} = 9z(2z - 1) - 4z = 3$. In case $z \geq \frac{1}{2}$, we see $4F_{13} \leq 3(-2z^2 + 2)(2z - 1) - 4z + 3 = -12z^2 + 14z - 3 \leq 4\Phi(z)$, and otherwise, we see $4F_{13} \leq 3(-z + 1)(2z - 1) - 4z + 3 = -6z^2 + 5z \leq 4\Phi(z)$.
We divide $\Delta^3$ into five parts.

If $y-4x \geq 0$, by (2.23), we have $4F_{17} = (1 - 3x)(3 - 3z) - 3z^2 + 4z - 1 \leq z(3 - 3z) - 3z^2 + 4z - 1 = -6z^2 + 7z - 1 \leq 4\Psi(z)$.

If $y-4x \leq 0$ and $x \leq \frac{1}{3}$, we have $z \leq \frac{1}{3}$, and by applying (2.23), we have $4F_{17} = (1 - 3x)(2 - 3z) - 3z^2 + 3z \leq z(2 - 3z) - 3z^2 + 3z \leq -6z^2 + 5z \leq 4\Psi(z)$.

When $x \geq \frac{1}{3}$, $y-4x \leq -1$, we have $4F_{17} = (3x - 1)(3z - 1) - 3z^2 + 3z$. If $z \geq \frac{1}{3}$, by applying (2.23), we have $(3x - 1)(3z - 1) - (z - 1)(3z - 1) = -3z^2 + 4z - 1$ and $4F_{17} \leq -6z^2 + 7z - 1 \leq 4\Psi(z)$. If $z \leq \frac{1}{3}$, we have $(3x - 1)(3z - 1) \leq 0$ and $4F_{17} \leq -3z^2 + 3z \leq 4\Psi(z)$.

If $y-4x \leq -1$, by applying (2.23), we have $4F_{17} = 3z(3x - 1) - 3z^2 + 2z \leq 3z(1 - 2z) - 3z^2 + 2z = -9z^2 + 5z \leq -6z^2 + 5z \leq 4\Psi(z)$.

The inequality of type 23. We have $4F_{23} = -6z^2 + 4z + (3y - 2)^* + (1 - 3x)^*$. If $(3y - 2)^* > 0$ and $(1 - 3x)^* > 0$, then $4F_{23} = -6z^2 + 7z - 1 \leq 4\Psi(z)$. Otherwise, by (2.24), we have $(3y - 2)^* + (1 - 3x)^* \leq z$ and $4F_{23} \leq -6z^2 + 5z \leq 4\Psi(z)$.

The inequality of type 27. We have $4F_{27} = -9z^2 + 8z - 9xz + (3y - 2)^*$. If $(3y - 2)^* = 0$, we have $-9xz \leq -3(z + 1)z$ and $4F_{27} \leq -6z^2 + 5z \leq 4\Psi(z)$. If $(3y - 2)^* > 0$, then $4F_{27} = -9z^2 + 11z + 3x(1 - 3z) - 2$. In case $z \leq \frac{1}{3}$, we have $4F_{27} \leq -9z^2 + 11z + (z + 1)(3 - 2z) = -6z^2 + 7z - 1 \leq 4\Psi(z)$.

In case $z \geq \frac{1}{3}$, we have $4F_{27} \leq -9z^2 + 11z + (z + 1)(1 - 3z) = -6z^2 + 7z - 1 \leq 4\Psi(z)$.

The inequality of type 47. We have $4F_{47} = -3z^2 + 3z \leq 4\Psi(z)$.

Thus (2.27) has been proved.

3. Proof of the Theorem

To prove (1.2), by (1.1), (2.1), and (2.15), it is enough to prove

$$\sup_{(x,y) \in \Delta} v(x,y) = \frac{130}{343}. \tag{3.1}$$

We prove that the above supremum is attained at

$$P_1 : \left( \frac{1}{7}, \frac{6}{7} \right).$$

The evaluation $v\left( \frac{1}{7}, \frac{6}{7} \right) = \frac{130}{343}$ can be found in [17]. Hence it suffices to prove

$$v(x, y) < v\left( \frac{1}{7}, \frac{6}{7} \right) = \frac{130}{343} \quad \text{for} \quad (x, y) \in \Delta \setminus \{P_1\}. \tag{3.2}$$
Write $\widetilde{v}_L(x, y) = -\widetilde{V}(\langle x \rangle, \langle y \rangle)$ and

$$\widetilde{v}_L(x, y) = -\widetilde{V}(\langle x \rangle, \langle y \rangle) + 2 \sum_{k=0}^{L} \frac{\widetilde{V}(\langle (2)^k x \rangle, \langle (2)^k y \rangle)}{(2)^k} \quad (L = 0, 1, 2, \ldots).$$

We shall prove

$$v(x, y) \leq \widetilde{v}_L(x, y) + 2^{-L} \frac{\Phi(x, y)}{7} \quad (L = 1, 0, 1, 2, \ldots). \quad (3.3)$$

Note that $v(x, y) = \widetilde{v}_L(x, y) + 2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \widetilde{V}(\langle (2^{L+j}x \rangle, \langle (2^{L+j}y \rangle, \langle x \rangle, \langle y \rangle) \quad (2^{L+j}x, \langle x \rangle, \langle y \rangle)$ and $\eta = (2^{L+j}x, \langle x \rangle, \langle y \rangle)$.

If $L + 3j + 1$ is even, we have by (2.27),

$$3 \sum_{j=1}^{\infty} \frac{\widetilde{V}(\langle (2)^{L+j}x \rangle, \langle (2)^{L+j}y \rangle, \langle x \rangle, \langle y \rangle)}{(2)^{L+j}} = \frac{F(\xi, \eta, x, y)}{2^{L+j+1}} \leq \frac{\Phi(x, y)}{2^{L+j+1}}.$$

If $L + 3j + 1$ is odd, by noting (2.19), we have

$$3 \sum_{j=1}^{\infty} \frac{\widetilde{V}(\langle (2)^{L+j}x \rangle, \langle (2)^{L+j}y \rangle, \langle x \rangle, \langle y \rangle)}{(2)^{L+j}} = \frac{F(\eta, \xi, x, y)}{2^{L+j+1}} \leq \frac{\Phi(x, y)}{2^{L+j+1}}.$$

Figure 2: The left triangle is $\Delta$. In $\Delta$, we take the quadrangles $\Delta_1\Delta_2: P_1P_2P_3P_4$, $\Delta_3\Delta_4: P_5P_6P_7P_8$, $\Delta_1\Delta_2: P_9P_{10}P_{11}P_{12}$, and $\Delta_3: P_{13}P_{14}P_{15}P_{16}$. In $\Delta_3$, we take the triangles $\Delta_3: P_{17}P_{18}P_{19}$.

In the sequel we split $\Delta$ in parts $\Delta \setminus \Delta_1 \setminus \Delta_2 \setminus \Delta_3 \setminus \Delta_4 \setminus \Delta_5 \setminus \Delta_6 \setminus \Delta_7 \setminus \Delta_8 \setminus \Delta_9 \setminus \Delta_9 \setminus \Delta_10 \setminus \Delta_11 \setminus \Delta_12 \setminus \Delta_13 \setminus \Delta_14 \setminus \Delta_15 \setminus \Delta_16 \setminus \Delta_17 \setminus \Delta_18 \setminus \Delta_19 \setminus \Delta_20$ and prove (3.2) for each part.
3.1. $\Delta \setminus \Delta_-$ part

We put

$$\Delta_{-1} = \{ (x, y) \in \Delta \mid \frac{65}{287} y - x \leq \frac{2}{7} \}, \quad \Delta_{-1^+} = \{ (x, y) \in \Delta \mid \frac{5}{7} y - x \leq \frac{222}{287} \},$$

and $\Delta_1 = \Delta_{-1} \cup \Delta_{-1^+}$, and prove

$$v(x, y) \leq v_{-1}(x, y) < \frac{130}{343} \quad \text{for} \quad (x, y) \in \Delta \setminus \Delta_-. \quad (3.4)$$

By (2.17) and (3.3), we have

$$v_{-1}(x, y) = \frac{16}{7} \Phi(x, y) = 2 - z + \frac{4}{7}(-12z^2 + 17z - 5) = \frac{41}{7}z^2 + \frac{61}{7}z - \frac{20}{7}$$

for $(x, y) \in \Delta$ with $\frac{2}{7} \leq y - x < 1$. In this case the inequality $v_{-1}(x, y) \geq \frac{130}{343}$ is equivalent to

When $\frac{2}{7} \leq y - x \leq \frac{2}{7}$, we have $v_{-1}(x, y) = \frac{41}{7}(z - \frac{49}{35})^2 + \frac{433}{178} < \frac{130}{343}$.

Note that $v_{-1}(x, y) = -(-z^2 + z) + \frac{4}{7} \Phi(z)$ is symmetric around $z = \frac{1}{2}$. Hence for $(x, y) \in \Delta$ with $0 \leq y - x \leq \frac{1}{2}$, the inequality $v_{-1}(x, y) \geq \frac{130}{343}$ is equivalent to $(x, y) \in \Delta_{-1}$. By combining these, we have proved (3.4).

Since $\Delta_{-1^+}$ is determined by $\frac{2}{7} \leq y - x \leq \frac{222}{287}$, $y + x \leq 1$, and $y + 2x \geq 1$, we see that it is the quadrangle with vertices $P_1$,

$$P_2 : \left( \frac{65}{574}, \frac{509}{574} \right), \quad P_3 : \left( \frac{65}{861}, \frac{731}{861} \right), \quad \text{and} \quad P_4 : \left( \frac{2}{21}, \frac{17}{21} \right).$$

Since $\Delta_{-1}$ is determined by $\frac{65}{287} \leq y - x \leq \frac{2}{7}$, $y + x \leq 1$, and $y + 2x \geq 1$, we see that it is the quadrangle with vertices

$$P_5 : \left( \frac{5}{14}, \frac{9}{14} \right), \quad P_6 : \left( \frac{5}{21}, \frac{11}{21} \right), \quad P_7 : \left( \frac{74}{287}, \frac{139}{287} \right), \quad \text{and} \quad P_8 : \left( \frac{111}{287}, \frac{176}{287} \right).$$

3.2. $\Delta_{-1^+} \setminus \Delta_1$ part

By noting (2.31), for $(x, y) \in \Delta_{-1}$, we have

$$\tilde{v}_1(x, y) = -3z^2 + z + (1 - 3x)^\gamma + (3y - 2)^\gamma. \quad (3.5)$$

We consider $\Delta_{-1^+}$. Since $y \leq \frac{2}{7}$ holds on $P_5$, $P_6$, $P_7$, and $P_8$, it holds on $\Delta_{-1^+}$. Hence for $(x, y) \in \Delta_{-1^+}$, we have $v_1(x, y) = -\frac{34}{21}z^2 + 2z + (1 - 3x)^\gamma$. Put

$$\Delta_1 = \{ (x, y) \in \Delta_{-1^+} \mid \frac{34}{21}y - \frac{113}{49}x + \frac{345}{343} \geq 0 \}.$$

On $(\Delta_{-1^+} \setminus \Delta_1) \cap \{ (x, y) \mid x \leq \frac{1}{2} \}$, by $-z^2 \leq -\frac{4}{7}z + \frac{4}{7}$, we have $v_1(x, y) = -\frac{130}{343}z^2 + 2z + (1 - 3x) \leq -\left( \frac{34}{21}y - \frac{113}{49}x + \frac{345}{343} \right) + \frac{130}{343} \leq \frac{130}{343}$. 13
We see \( P_6, P_7 \in \Delta_1 \) and \( P_5, P_8 \notin \Delta_1 \). Hence \( \Delta_1 \) is determined by \( \frac{65}{27} \leq y - x \leq \frac{2}{7}, \) \(-\frac{34}{99} y - \frac{113}{99} x + \frac{345}{337} \geq 0, \) and \( y + 2x \geq 1, \) which is the quadrangle with vertices \( P_6, P_7, \)

\[
P_9 : \left( \frac{1705}{6027} \cdot 3070, \right) \text{, and } P_{10} : \left( \frac{277}{1029} \cdot 571 \right)
\]

We can verify \( x < \frac{1}{4} \) on \( P_6, P_7, P_9, \) and \( P_{10}, \) and hence \( x < \frac{1}{4} \) on \( \Delta_1 \). Therefore we see \( \Delta_{-1} \cap \{ (x, y) | x = \frac{1}{4} \} \) is outside of \( \Delta_1 \) and \( v_1(\frac{1}{4}, \frac{1}{4} + c) < \frac{130}{343} \) for \( \frac{65}{27} \leq c \leq \frac{2}{7}. \) If we consider on the line \( y - x = c, \) we see \( v_1(x, y) = v_1(x, x + c) = v_1(\frac{1}{4}, \frac{1}{4} + c) \) for \( x \geq \frac{1}{4} \), and consequently we have \( v_1(x, y) < \frac{130}{343} \) for \( (x, y) \in \Delta_{-1} \cap \{ (x, y) | x \geq \frac{1}{4} \} \). Hence we have proved

\[
v(x, y) \leq v_1(x, y) < \frac{130}{343} \text{ for } (x, y) \in \Delta_{-1} \setminus \Delta_1.
\]

(3.6)

Since we have \( y \leq \frac{2}{7} \) in \( \Delta_{-1} \) and \( \Delta_1 \subset \Delta_{-1}, \) we have \( y \leq \frac{2}{7} \) in \( \Delta_1. \) We have also verified \( x < \frac{1}{4} \) in \( \Delta_1. \)

3.3. \( \Delta_1 \) part

We consider \( \Delta_1. \) We can verify \( P_6, P_7, P_9, \) and \( P_{10} \) satisfy \( 0 < 4x - y < 1, 1 < 4y - x < 2, \) \( 0 < 3 - 9x < 1, 0 < 5 - 9y < 1, 0 < 3 - 8x - y < 1, \) and \( 0 < 5 - 8y - x < 1. \) Hence we see that these inequalities hold in \( \Delta_1. \) We have \( \tilde{V}_1(x, y) = -3z^2 + z + (1 - 3x) \) by (3.5) and \( \tilde{V}(4x, 4y, (x), (y)) = -4z^2 + z \) by (2.32). By (2.8) we have

\[
-\tilde{V}((-8x), (-8y), (x), (y)) = -V(5 - 8y - x, z) + V(3 - 9x, z)
\]

\[
= -(z - (5 - 8y - x)z) + (z - (3 - 9x)z) = -8z^2 + 2z.
\]

Hence by (2.23), \( v_3(x, y) = -\frac{52}{7} z^2 + \frac{29}{4} z + (1 - 3x) \leq -\frac{52}{7} z^2 + \frac{13}{4} z = -\frac{52}{7} \left( z - \frac{7}{52} \right)^2 + \frac{91}{256} \leq \frac{91}{256} < \frac{130}{343}. \)

Therefore,

\[
v(x, y) \leq v_3(x, y) < \frac{130}{343} \text{ for } (x, y) \in \Delta_1.
\]

(3.7)

3.4. \( \Delta_{-1} \setminus \Delta_3 \) part

In \( \Delta_{-1}, \) we have \( x < \frac{1}{4} \) and \( y \geq \frac{2}{7}. \) Hence by (3.5), we have \( \bar{V}_1(x, y) = -3z^2 + 4z - 1. \)

We can verify \( P_1, P_2, P_3, \) and \( P_4 \) satisfy \( 4x - y < 0, 3 < 4y - x, 0 < 8 - 9y < 1, \) and \( 0 \leq 2 - 8x - y < 1. \) Hence these inequalities holds on \( \Delta_{-1}. \) By (2.32), we have \( \bar{V}(4x, 4y, (x), (y)) = -4z^2 + 7z - 3. \) By (2.4), (2.8), and (2.10), we have

\[
-\bar{V}((-8x), (-8y), (x), (y)) = \bar{V}((-8x), (-8y), (x), (x))
\]

\[
= -V(2 - 8x - y, 1 + x - y) + V(8 - 9y, 1 + x - y)
\]

\[
= -(2 - 8x - y)z - (1 - 9x)^+ + (8 - 9y)z - (7 - 8y - x)^+
\]

\[
= -8z^2 + 6z + (1 - 9x)^+ - (7 - 8y - x)^+.
\]

Hence in \( \Delta_{-1}, \) we have

\[
\bar{v}_3(x, y) = -7z^2 + 9z - \frac{5}{2} + \frac{1}{4}(1 - 9x)^+ - \frac{1}{4}(7 - 8y - x)^+.
\]

(3.8)
and \( v_3(x, y) = \frac{1}{35}(-208z^2 + 269z - 75 + 7(1 - 9x)^* - 7(7 - 8y - x)^*) \).

Firstly, in \( \{(x, y) \in \Delta_{-1} \mid x \leq \frac{1}{4}\} \), by \((-7 - 8y - x)^* \leq -(7 - 8y - x) \), we have \( v_3(x, y) \leq \frac{1}{35}(-208z^2 + 325z - 117) =: v_3^*(z) \). Since \( v_3^*(z) \) increasing in \( z \in (-\infty, \frac{123}{196}) \), by \( \frac{123}{196} < \frac{122}{196} \) and by noting \( z \leq \frac{123}{196} \) in \( \Delta_{-1} \), we have \( v_3(x, y) \leq v_3^*(\frac{123}{196}) = \frac{411805}{340947} < \frac{120}{197} \).

Secondly, on \( \{(x, y) \in \Delta_{-1} \mid x \geq \frac{1}{4}\} \), we have \( v_3(x, y) = \frac{1}{35}(-208z^2 + 269z - 75) - \frac{1}{4}(7 - 8y - x)^* \).

On \( \{(x, y) \in \Delta_{-1} \mid x \geq \frac{1}{4}, y - x > \frac{1005}{1379}\} \), by (2.26) and \((-7 - 8y - x)^* \leq 0 \), we have \( v_3(x, y) \leq -\frac{197}{196}z + \frac{1525}{1379} - \frac{197}{196}(z - \frac{1005}{1379}) + \frac{123}{196}x - \frac{349}{343} \). On \( \{(x, y) \in \Delta_{-1} \mid x \geq \frac{1}{4}, y - x \leq \frac{1005}{1379}\} \), by (2.26) and \((-7 - 8y - x)^* \leq -(7 - 8y - x) \), we have \( v_3(x, y) \leq \frac{1005}{196}z + \frac{123}{196}x - \frac{349}{343} + \frac{122}{196} < \frac{130}{197} \). Let

\[
\Delta_3 := \left\{(x, y) \mid y + x \leq 1, \quad 5 \leq y - x \leq \frac{1005}{1379}, \quad \frac{195}{196}y + \frac{123}{98}x - \frac{349}{343} \geq 0 \right\}.
\]

Since \( \Delta_3 \) is the quadrangle having vertices \( P_1 \),

\[
P_{11} : \left(\frac{187}{1379}, \frac{1192}{1379}\right), \quad P_{12} : \left(\frac{1613}{12411}, \frac{10658}{12411}\right), \quad \text{and} \quad P_{13} : \left(\frac{421}{3087}, \frac{2626}{3087}\right),
\]

we can verify \( P_1, P_{11}, P_{12}, P_{13} \in \Delta_{-1} \), and \( \Delta_3 \subset \Delta_{-1} \). Combining these, we have

\[
v(x, y) \leq v_3(x, y) < \frac{130}{343} \quad \text{for} \quad (x, y) \in \Delta_{-1} \setminus \Delta_3.
\]

\[\text{(3.9)}\]

3.5. \( \Delta_3 \setminus \Delta_3 \) part

We consider \( \Delta_3 \). By calculating values at \( P_1, P_{11}, P_{12}, \) and \( P_{13} \), we have \( 1 - 9x < 0, 0 < 15y - 12 < 1, 0 < 16x - y - 1 < 1, 0 < 16y - x - 13 < 1, 0 < -33x + 5 < 1, 0 < -33y + 29 < 1, 0 < -32x - y + 6 < 1, \) and \( 0 < -32y - x + 28 < 1 \) on \( \Delta_3 \). By (3.8), we have \( v_3(x, y) = -7z^2 - 9z - \frac{1}{2}(7 - 8y - x)^* \). By (2.4) and (2.8), we have

\[
\bar{V}((16x), (16y), (x), (y)) = \bar{V}((16x), (16y), (y), (x)) = V((16x) - y - 1, 1 + x - y) - V(15y - 12, 1 + x - y) = ((16x - y - 1)z - (15x - 2)^*) - ((1 + x - y) - (15y - 12)(1 + x - y)) = -16z^2 + 12z + 15y - 13 - (15x - 2)^*,
\]

and

\[
- \bar{V}((-32x), (-32y), (x), (y)) = -V(-32y - x + 28, y - x) + V(-33x + 5, y - x) = -((-32y - x + 28) - (-32y - x + 28)z) + ((-33x + 5) - (-33x + 5)z) = -32z^2 + 55z - 23.
\]

Hence we have

\[
\bar{v}_5(x, y) = -11z^2 + \frac{223}{16}z - \frac{89}{16} - \frac{1}{4}(7 - 8y - x)^* + \frac{1}{8}(15y - (15x - 2)^*),
\]

and by \((-15x - 2)^* \leq -(15x - 2) \) we have \( v_5(x, y) \leq -\frac{311}{28}z^2 + \frac{447}{28}z - \frac{75}{14} - \frac{1}{4}(7 - 8y - x)^* \).

On \( \{(x, y) \in \Delta_3 \mid y - x > \frac{1574}{2177}\} \), by \( -\frac{1}{4}(7 - 8y - x)^* \leq 0 \), we have \( v_5(x, y) \leq -\frac{311}{28}z^2 + \frac{447}{28}z - \frac{75}{14} = -\frac{311}{28}(z - \frac{75}{14})(z - \frac{130}{2177}) + \frac{330}{2177} \). Note that \( \frac{330}{2177} < \frac{170}{197} \).
On \([x, y) \in \Delta_3\ \| 411y + 15x - 624 < 0\), by (2.26) and \(-7 - 8y - x^* \leq -7 - 8y - x\), we have \(v_3(x, y) \leq (411y + 15x - 624 + 130) \leq 130\). Let 

\[
\Delta_3 := \left\{ (x, y) \mid y + x \leq 1, \quad \frac{5}{7} \leq y - x \leq \frac{1574}{2177}, \quad \frac{411}{196} y + 15x - 624 \frac{343}{130} \geq 0 \right\}.
\]

It is the triangle with vertices \(P_1\).

\(P_{14} : \left[ \frac{603}{4354}, \frac{3751}{4354} \right], \quad \text{and} \quad P_{15} : \left[ \frac{43114}{320019}, \frac{274492}{320019} \right].\)

\(P_{14}\) and \(P_{15}\) are located in \(\Delta_3\), and we see \(\Delta_5 \subset \Delta_3\). We have proved 

\[
\nu(x, y) \leq v_5(x, y) < \frac{130}{343} \quad \text{for} \quad (x, y) \in \Delta_3 \setminus \Delta_5, \quad (3.10)
\]

3.6. \(\Delta_5 \setminus \Delta_6\) \(\)part

We consider \(\Delta_5\). We can show that \(0 < 15x - 2, 0 < 64y - x - 54 < 1, 0 < 63x - 8 \leq 1, \) and \(0 \leq 63y - 54 < 1\) hold at \(P_1, P_{14}\), and \(P_{15}\), and hence on \(\Delta_5\). We have \(v_5(x, y) = -11z^2 + \frac{283}{1568} \frac{z}{16} - \frac{1}{1574} - \frac{1}{7 - 8y - x}^*\). By (2.8) we have 

\[
\tilde{V}(64x, 64y, (x), (y)) = V(64y - x - 54, z) - V(z, 63x - 8) = z - z(64y - x - 54) - z(1 - 63x + 8) + (y - 64x + 8)^* = -64z^2 + 46z + (y - 64x + 8)^*.
\]

Therefore we have 

\[
\tilde{v}_6(x, y) = -11z^2 + \frac{69}{4} z - \frac{85}{16} - \frac{1}{4} (7 - 8y - x)^* + \frac{1}{32} (-64y + x + 8)^*.
\]

and by \(-7 - 8y - x)^* \leq 0\) we have \(v_6(x, y) \leq -\frac{711}{1568} z^2 + \frac{3881}{1568} z - \frac{1105}{128} + \frac{1}{32} (-64y + x + 8)^*\). We show that \((x, y) \in \Delta_5\) and \(y - x \geq \frac{1035}{1574}\) imply \(v_6(x, y) \leq \frac{130}{343}\). We divide \(\Delta_5\) into two parts.

In \([x, y) \in \Delta_5\ \| -64x + y + 8 < 0\), by (2.26), we have \(v_6(x, y) \leq -\frac{224}{1568} z^2 + \frac{14545}{10976}\), and hence we have \(v_6(x, y) < \frac{130}{343}\) if \(y - x > \frac{1035}{1574}\).

In \(\Delta_5^* := \{ x, y \in \Delta_5 \mid -64x + y + 8 \geq 0 \},\) by (2.26), we have \(v_6(x, y) \leq -\frac{253}{196} - \frac{1063}{1574} x + \frac{13128}{1574} := v_6^*(x, y).
\]

By \(P_{14}, P_{16} \notin \Delta_6^*\) and \(P_{15} \in \Delta_6^*\), we find that \(\Delta_6^*\) is the triangle with vertices \(P_{13}\).

\(P_{16} : \left( \frac{2110}{15239}, \frac{13128}{15239} \right), \quad \text{and} \quad P_{17} : \left( \frac{4252}{30723}, \frac{26344}{30723} \right).\)

We can verify that \(v_6^*(x, y) < \frac{130}{343}\) holds on these points, and hence holds on \(\Delta_6^*\). Set 

\[
\Delta_6 := \left\{ (x, y) \mid y + x \leq 1, \quad \frac{5}{7} \leq y - x \leq \frac{10385}{14511}, \quad \frac{411}{196} y + 15x - 624 \frac{343}{130} \geq 0 \right\}.
\]

By \(\frac{10385}{14511} < \frac{1574}{2177}\), we see \(\Delta_5 \subset \Delta_5\) and, we have seen that 

\[
\nu(x, y) \leq v_6(x, y) < \frac{130}{343} \quad \text{for} \quad (x, y) \in \Delta_5 \setminus \Delta_6. \quad (3.11)
\]

It is the triangle with vertices \(P_{18}\).

\(P_{18} : \left( \frac{2063}{14511}, \frac{12448}{14511} \right), \quad \text{and} \quad P_{19} : \left( \frac{301991}{2133117}, \frac{1828586}{2133117} \right).\)
3.7. $\Delta_6$ part

We consider $\Delta_6$. We can show that $-64x+y+8 < 0, 0 < -129x+19 < 1, 0 < -128y-x+110 < 1, -129y + 110 < 0, 0 < 255x - 36 < 1, 0 < 256y - x - 219 < 1, 255y - 219 < 0$, and $256x - y - 36 < 0$ hold on $P_1$, $P_{18}$, and $P_{19}$, and hence hold on $\Delta_6$. We have the estimate $v_6(x,y) = -13z^2 + \frac{69}{2}z - \frac{65}{16} - \frac{1}{3}(7 - 8y - x)^+ \leq -13z^2 + \frac{69}{2}z - \frac{65}{16}$. By (2.8) we have

$$V((-128x), (-128y), (y), (y)) = -V(-128y - x + 110, z) + V(-129x + 19, z) = -(-128y - x + 110) + (-128y - x + 110)z + (-129x + 19)(1 - z) - (-128x - y + 19)^+ \leq -128z^2 + 219z - 91,$$

$$V ((256x), (256y), (x), (y)) = V(256y - x - 219, z) - V(255x - 36, z) = (256y - x - 219) - (256y - x - 219)z - (255x - 36) + (255x - 36)z = -256z^2 + 439z - 183.$$

We therefore have $v_6(x,y) \leq -17z^2 + \frac{3085}{125}z - \frac{1045}{125}$, and by (2.26) we have

$$v_6(x,y) \leq -\frac{3811}{224}z^2 + \frac{5403}{224}z - \frac{915}{112} \leq \frac{289}{1568}(\frac{5}{7} - z) + \frac{130}{343} < \frac{130}{343} \text{ for } (x,y) \in \Delta_6 \setminus \{P_1\}. \quad (3.12)$$

By combining (3.4), (3.6), (3.7), (3.9), (3.10), (3.11), and (3.12), we have (3.2).

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**References**


