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<td>著者 (Author(s))</td>
<td>Fukuyama, Katusi</td>
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<tr>
<td>掲載誌・巻号・ページ (Citation)</td>
<td>Sugaku Expositions, 25(2): 189-207</td>
</tr>
<tr>
<td>刊行日 (Issue date)</td>
<td>2012</td>
</tr>
<tr>
<td>資源タイプ (Resource Type)</td>
<td>Journal Article / 学術雑誌論文</td>
</tr>
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<td>版区分 (Resource Version)</td>
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<td>URL</td>
<td><a href="http://www.lib.kobe-u.ac.jp/handle_kernel/90003859">http://www.lib.kobe-u.ac.jp/handle_kernel/90003859</a></td>
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PDF issue: 2019-01-11
Limit theorems for lacunary series and the theory of uniform distribution

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November 6, 2010

1 Introduction

According to Weierstrass [90], Riemann mentioned in his lecture on 1861 that the continuous function defined by \( \sum \frac{\sin k^2 x}{k^2} \) is nowhere differentiable \(^1\), which is a famous story frequently referred as the first moment when lacunary series appeared. Weierstrass [91] used a lacunary series to construct a power series whose convergence circle is the natural boundary.

There are already detailed surveys by Kahane[56], Murai [67, 68], Kawata [57] which explain the history of various studies on lacunary series.

In this article, we specially focus on relationship among probabilistic behavior of lacunary series \( \sum f(n_k x) \) and the uniform distribution theory, and try to state the recent development of studies.

2 Limit Theorems for lacunary trigonometric series

When \( n_k \) diverges very rapidly, behavior of a series \( \sum k a_k \cos 2\pi n_k x \) is very typical. If the Hadamard’s gap condition

\[
n_{k+1}/n_k \geq q > 1
\]

is satisfied, the series converges almost everywhere (Kolmogorov [62]) or diverges almost everywhere (Zygmund [93]) according as \( \sum a_k^2 \) converges or diverges. It imitates the behavior of sum of independent random variables.

It had been a problem asking if the series obeys the central limit theorem, which is a more classical limit theorem than convergence theorems. The first result of this kind was proved by Kac [50]:

**Theorem 1** If a sequence \( \{n_k\} \) of positive integers has large gaps

\[
n_{k+1}/n_k \rightarrow \infty,
\]

\(^1\)Riemann’s lacunary series has very small gaps and is very difficult to investigate its property. Weierstrass gave up the proof of its non-differentiability and constructed a nowhere differentiable function by using lacunary series having larger gaps. That is so-called the Weierstrass function.

Riemann’s function was studied later extensively, and Hardy [44] proved that it is non-differentiable at any irrational multiple of \( \pi \). Gerver [41, 42] proved that this function has differentiation coefficient \(-1/2\) at any point of the form \( \pi \) times ratio of odd integers, and non-differentiable elsewhere. Hence the conjecture by Riemann was negatively solved.
the central limit theorem

$$\lim_{N \to \infty} \left\{ x \in [0, 1] \left| \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \sqrt{2} \cos 2\pi n_k x \leq a \right. \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-u^2/2} \, du$$

holds. Here $|A|$ denotes the Lebesgue measure of a measurable set $A$.

The problem to prove the central limit theorem holds under the Hadamard’s gap condition in the same way as the convergence theorems, was extensively studied.

It was said that Erdős succeeded in proving but the result was not published. Ferrand–Fortet [29] announced only the result, but the proof was not published, and the first result succeeded in proving rigorously was given by Salem–Zygmund [79].

Erdős [27] tried to weaken the gap condition, and proved the central limit theorem under the gap condition

$$n_{k+1}/n_k > 1 + c_k/\sqrt{k}, \quad (c_k \to \infty),$$

which we call Erdős’ gap condition. He also gave an example for which the central limit theorem fails to hold under the condition which we call Erdős’ gap condition. He also gave an example for which the central limit theorem does not obey the central limit theorem.

The problem was open and Murai [66] partially solved. We here state his result.

First, we introduce notation. Denote by $\# A$ the cardinality of $A$. For $G \subset \mathbb{N}$, put

$$H(G) = \left\{ (m_1, m_2, m_3, m_4) \in (G \cup -G)^4 \left| m_1 + m_2 + m_3 + m_4 = 0, |m_i| \neq |m_j| (i \neq j) \right. \right\}.

**Theorem 2 (Murai [66])**

(i) If a sequence $\{n_k\}$ of positive integers satisfies the following two conditions, then the central limit theorem holds.

$$\#(\{n_k\} \cap (2^L, 2^{L+1}]) = O(\#(\{n_k\} \cap (0, 2^L])^\alpha), \quad (0 \leq \alpha < 1), \quad \text{(3)}$$

$$\# H(\{n_1, \ldots, n_k\}) = o(k^2), \quad \text{(4)}$$

(ii) If $\gamma > 4/9$ then the Erdős’ sequence $n_k = [\exp(k^\gamma)]$ satisfies (3) and

$$\# H(\{n_1, \ldots, n_k\}) = o(k^{2-\delta}), \quad (\delta > 0), \quad \text{(5)}$$

and hence the central limit theorem holds.

---

2Erdős only stated an example and didn’t give a proof that it does not obey the central limit theorem. Here we gave a brief explanation about the proof.

The proof of the central limit theorem in this paper is by proving convergence of moments, i.e., $E(S_N/\sqrt{N})^k \to EG^k$ ($N \to \infty$). Here $S_N$ denotes the $N$-th partial sum of the lacunary trigonometric series and $G$ denotes the standard normal random variable.

By following the above proof by assuming a weaker condition $c_k = c > 0$ in gap condition, we can prove $E(S_N/\sqrt{N})^k = O(1)$ ($N \to \infty$). Especially we see that $S_N/\sqrt{N}$ is bounded sequence in $L^k$ space, and hence by uniform integrability we must have $E(S_N/\sqrt{N})^k = EG^k + o(1)$ if we assume that the central limit theorem holds. On the other hand, in case of the example Erdős presented, we have $E(S_N/\sqrt{N})^k \geq C (C > 0)$, which shows that the example never obeys the central limit theorem.
The condition (3) is very weak condition, which is implied by the gap condition
\[ \frac{n_{k+1}}{n_k} > 1 + ck^{-\alpha}, \quad (c > 0, \ 0 \leq \alpha < 1). \]  
(6)
The Erdős’ sequence satisfies this condition. The condition (4) is implied by Erdős’ gap condition and is not necessary to assume when \( \alpha < 1/2 \). On the other hand, it appeared to be necessary to assume when gap condition is weaker.

As to the question asking how small gap sequence satisfying the central limit theorem can exists, we have the following result.

**Theorem 3 (Berkes [4])** For any sequence \( L_k \to \infty \), there exists a sequence \( \{n_k\} \) of integers satisfying \( 0 < n_{k+1} - n_k = O(L_k) \).

It is proved by randomization, in other word, by Erdős’ method. First one construct a probability measure on a space of sequences satisfying the required condition, and then prove that the central limit theorem holds with probability one, and conclude that there exists at least one sequence satisfying the required condition and obeying the central limit theorem. Consequently, one cannot decide if the central limit theorem holds for given concrete sequence, and the assertion is said to be an existence theorem.

We have important examples of sequences which have smaller gaps than Hadamard’s. These are the Hardy-Littlewood-Pólya sequences. Let \( \tau \geq 2 \) and \( q_1, \ldots, q_\tau \geq 2 \) are positive integers relatively prime from each other. The arrangement in increasing order of the set
\[ \{q_1^{m_1} \cdots q_\tau^{m_\tau}\}_{m_1, \ldots, m_\tau \geq 0} \]
is called as Hardy-Littlewood-Pólya sequence. Although Tijdeman [89] proved that this sequence satisfies the gap condition
\[ \frac{n_{k+1}}{n_k} > 1 + ck^{-\alpha}, \quad (c > 0, \ \alpha > 0), \]
it is unknown if this \( \alpha \) can be take to be less than 1. But one can be verify that this sequence satisfies the conditions (3) and (4), and the sum \( \sum \cos 2\pi n_k x \) obeys the central limit theorem.

We now move to the topic of another typical limit theorem, the law of the iterated logarithm
\[ \limsup_{N \to \infty} \frac{1}{\sqrt{2N \log \log N}} \sum_{k=1}^{N} \sqrt{2} \cos 2\pi n_k x = 1, \quad \text{a.e. } x. \]

By assuming the Hadamard’s gap condition (1), Salem-Zygmund [80] proved that the left hand side is bounded by 1, and Erdős-Gál [25] proved the equality. Takahashi [86, 87] weakened the gap condition to Takahashi’s gap condition
\[ \frac{n_{k+1}}{n_k} > 1 + ck^{-\alpha}, \quad (c > 0, \ \alpha < 1/2) \]  
(7)
They are called Hardy-Littlewood-Féjer sequences in [14] and [76, 19]. Both of these names are due to Walter Philipp.

\[ \text{The gap condition Takahashi assumed frequently is} \]
\[ \frac{n_{k+1}}{n_k} > 1 + ck^{-\alpha}, \quad (c > 0, \ \alpha \leq 1/2) \]
and concluded the central limit theorem for weighted sums \( \sum_{k=1}^{N} a_k \cos(2\pi n_k x) / A_N \) where \( A_N^2 = (a_1^2 + \cdots + a_N^2)/2 \) by assuming \( a_k = o(A_k k^{-\alpha}) \) or some stronger conditions. In this article, we restrict ourselves to the case when \( a_k = \sqrt{2} \) which consequently implies \( \alpha < 1/2 \). That why we call Takahashi’s gap condition when we assume \( \alpha < 1/2 \).
which is slightly stronger than the Erdős’ gap condition. Prior to the Murai’s result we mentioned before, Berkes (1978) assumed (6) and

$$\#H(\{n_l, \ldots, n_k\}) = O(k^\delta(k-l)^2), \quad (0 < \delta < 1 - \alpha)$$

and proved the almost sure invariance principles, and as Corollaries of this, derived the central limit theorem and law of the iterated logarithm. But this condition is too strong to be verified for the Erdős’ sequence.

It is known that we have the almost sure invariance principles and consequently the law of the iterated logarithm, if we assume (6) and (5) which is stronger than Murai’s condition and weaker than Berkes’ condition ([34]). By this we see that the law of the iterated logarithm holds for the Erdős’ sequence with $$\gamma > 4/9$$.

Berkes [6, 7, 8, 10] proved the law of the iterated logarithm under Erdős’ gap condition with $$C(\log \log k)^\gamma \leq c_k (C > 0, \gamma > 1/2)$$, and proved the existence of counterexample if $$\gamma = 1/2$$. As to the central limit theorem Erdős’ gap condition itself is a best possible condition, we see the difference among the domains where the central limit theorem or the law of the iterated logarithm holds.

As to the law of the iterated logarithm for lacunary trigonometric series with small gaps, Berkes [4] proved the same existence theorem as the central limit theorem.

Philipp [75] proved the almost sure invariance principles for lacunary trigonometric series whose frequencies are given by Hardy-Littlewood-Pólya sequence, and derived the law of the iterated logarithm. It was used for asymptotic estimate for discrepancies which will be stated later.

3 The law of the iterated logarithm for lacunary series

Until 1933 when Kolmogorov published the celebrated ‘Grundbegriffe der Wahrscheinlichkeitsrechnung’ [61], it was not trivial whether one can define a sequence of random variables with required property on some space. Among many efforts, we can find a method proposed by Steinhaus to consider the space $[0, 1]$ equipped with Lebesgue measure and regard functions on that space as random variables. On this probability space, Kac proved some results which are basic theorems in contemporary probability theory. For example, Kac[47] proved that $$X_1, \ldots, X_n$$ are independent if and only if

$$\exp(it_1X_1 + \cdots + it_nX_n) = \exp(it_1X_1) \cdots \exp(it_nX_n)$$

for every $$t_1, \ldots, t_n$$. In this historical context, lacunary series $$\sum f(n_kx)$$ was studied before the studies of the central limit theorem for lacunary trigonometric series. From now on we will be concerned with the following central limit theorem and the law of the iterated logarithm:

$$\lim_{N \to \infty} \left\{ x \in [0, 1] \left| \frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(n_kx) \leq a \right\} \right. = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{0} e^{-u^2/2\sigma^2} du; \quad (8)$$

$$\limsup_{N \to \infty} \frac{1}{\sqrt{2N \log \log N}} \sum_{k=1}^{N} f(n_kx) = \sigma, \quad \text{a.e. } x. \quad (9)$$

Here and after we assume that $$f$$ is a real valued function satisfying

$$f(x+1) = f(x), \quad \int_{0}^{1} f(x) \, dx = 0, \quad \|f\|_2^2 = \int_{0}^{1} |f(x)|^2 \, dx < \infty. \quad (10)$$
The first result on this matter is the next theorem by Kac [48].

**Theorem 4** Let \( f \) be a \( \beta \)-Lipschitz (\( \beta > 0 \)) real valued function satisfying (10) and let \( n_k = 2^{k(k-1)/2} \). Then the central limit theorem (8) holds and the limiting variance is given by \( \sigma^2 = \|f\|^2 \).

One year later Kac [49] proved the central limit theorem for \( f(x) = x - \lfloor x \rfloor - 1/2 \) and \( n_k = 2^k \), and found that the value of \( \sigma \) is not equal to \( \|f\|_2 \) in this case. After Theorem 4, Fortet [31] investigated the sequence \( n_k = 2^k \) and proved the central limit theorem and the law of the iterated logarithm. The proof was based on the argument of Markov chain. Kac did not understand the proof, and especially pointed confusion in the case of function of bounded variation. The following result gave a rigorous proof on this result.

**Theorem 5 (Kac [51])** Suppose that \( f \) is a real valued function satisfying (10), and that \( f \) is a \( \beta \)-Lipschitz function (\( \beta > 0 \)) or a function of bounded variation. For the sequence \( n_k = 2^k \), the central limit theorem (8) holds and the limiting variance \( \sigma^2 \) is given as below:

\[
\sigma^2 = \int_0^1 f^2(x) \, dx + 2 \sum_{k=1}^{\infty} \int_0^1 f(x) f(2^k x) \, dx.
\]

In this theorem, we can see that the limiting variance depends strongly on \( f \) and we can find dependence as a stationary sequence. It is a great difference from the case of trigonometric series.

The law of the iterated logarithm corresponding to this result is given as below:

**Theorem 6 (Maruyama [65])** Let \( f \) be a \( \beta \)-Lipschitz function (\( \beta > 0 \)) satisfying (10). Then for the sequence \( n_k = 2^k \), the law of the iterated logarithm (9) holds with \( \sigma^2 \) given by (11).

In this theorem, a function of bounded variation is missing which was included in the case of the central limit theorem. Izumi [46] assumed the boundedness of \( f \) and weakened the regularity condition to the following one:

\[
\int_0^1 \max_{0 \leq v \leq u} |f(x+v) - f(x)| \, dx = O((\log 1/u)^{-1-\delta}), \quad (\delta > 0).
\]

If \( f \) is of bounded variation we see that left hand side is of \( O(u)^5 \) and see that the law of the iterated logarithm holds. Hence the missing part in the theorem of Maruyama was compensated by this theorem.

From the contemporary point of view, it is natural to regard these results as consequence of the uniform mixing property of binary transform, and it seems to be difficult to expect the same results for every rapidly diverging \( \{n_k\} \). But they expected the same limit theorems under the Hadamard’s gap condition, until the following counterexample was constructed by Erdős-Fortet [52, 54]. For \( f(x) = \cos 2\pi x + \cos 4\pi x \) and \( n_k = 2^k - 1 \), it holds that

\[
\lim_{N \to \infty} \left\{ x \in [0, 1] \left| \frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(n_k x) \leq a \right. \right\} = \int_0^1 \frac{dx}{\sqrt{4\pi \cos^4 \pi x}} \int_{-\infty}^{a} e^{-u^2/4\cos^2 \pi x} \, du,
\]

\footnote{If \( f \) is of bounded variation, it equals almost everywhere to the indefinite integral of signed measure \( \nu \) with period \( 1 \). For \( 0 \leq v \leq u \) we see that \( |f(x+v) - f(x)| \) is bounded by \( \int 1_{(x,x+u)}(t)|\nu|(dt) \), and we see that the left hand side of (3) is bounded by \( \int_0^1 \int (x,x+u) (t)|\nu|(dt) = u|\nu|([0,1]) \).}
i.e., the central limit theorem fails to hold, and for the law of the iterated logarithm, it fails to hold as below:

\[
\limsup_{N \to \infty} \frac{1}{\sqrt{2N \log \log N}} \sum_{k=1}^{N} f(n_k x) = \sqrt{2} \lvert \cos \pi x \rvert, \quad \text{a.e. } x.
\]

By this example, it is known that the Hadamard’s gap condition is not sufficient to have the central limit theorem and the law of the iterated logarithm for \( \sum f(n_k x) \) with general function \( f \), and it is known to be necessary to assume stronger gap condition or some structure on the sequence \( \{n_k\} \).

By strengthening the gap condition to the large gap condition (2) and assuming \( f \) to be \( \beta \)-Lipschitz function (\( \beta > 0 \)) satisfying (10), Kac [50] proved the central limit theorem (8) with \( \sigma = \|f\|_2 \), and Takahashi [85] proved the corresponding law of the iterated logarithm (9).

Later Takahashi [83] weakened \( \beta \)-Lipschitz regularity (\( \beta > 0 \)) to \(^6\)

\[
\|f(\cdot + h) - f(\cdot)\|_2 = O((\log 1/h)^{1-\delta}), \quad (\delta > 0)
\]

and proved the central limit theorem (8). This condition satisfied by functions of bounded variation, since \( \|f(\cdot + h) - f(\cdot)\|_2 = O(h^{1/2}) \). \(^7\) We know that the law of the iterated logarithm (9) can be proved under very weak condition below ([37]):

\[
\|f(\cdot + h) - f(\cdot)\|_2 = O((\log 1/h)^{-2}(\log \log 1/h)^{-\alpha}) \quad (\alpha > 0).
\]

For arbitrary \( \theta > 1 \), we can prove a similar result for the sequence \( n_k = \theta^k \) as Kac proved in the case of powers of two. The sum \( \sum f(\theta^k x) \) appearing here is called Riesz-Raikov sum. \(^8\) In case when \( \theta \) is not integer, \( \theta^k x \) cannot be regarded as a result of iteration of the transform \([0, 1) x \mapsto \theta x \) (mod 1), which is completely different from the situation of \( 2^k x \).

Petit [70] assumed (10) on \( \beta \)-Lipschitz (\( \beta > 1/2 \)) function \( f \), and proved (8) for algebraic \( \theta > 1 \). For algebraic \( \theta \) which is not a power root of rationals, one has \( \sigma = \sigma(\theta, f) = \|f\|_2 \), and when \( \theta \) is given by

\[
r = \min\{k \in \mathbb{N} : \theta^k \in \mathbb{Q}\} \quad \theta^r = p/q, \quad p, q \in \mathbb{N}, \quad \gcd(p, q) = 1
\]

one has

\[
\sigma^2 = \sigma^2(\theta, f) = \int_0^1 f^2(x) \, dx + 2 \sum_{k=1}^{\infty} \int_0^1 f(\theta^k x) f(q^k x) \, dx.
\]

\(^6\)Actually Takahashi [83] assumed \( \|f - S_{f,n}\|_2 = O((\log n)^{-1-\delta}) \) which is equivalent to (12). Here \( S_{f,n} \) is the \( n \)-th subsum of the Fourier series of \( f \). This equivalence is proved by an inequality \( \|f - S_{f,n}\|_2 \leq C_1 \omega^{(2)}(1/N, f) \) which can be found as (3.3) in Zygmund [94] 241pp, and an inequality \( \omega^{(2)}(1/N, f) \leq C_2 \sum_{k=0}^{\infty} \|f - S_{f,k}\|_2/N \) which is (2.6) in Bari [2] 160pp. Hereafter, some of the conditions are translated to the equivalent ones by using these inequalities.

\(^7\)As in the same way we presented in footnote 5), we have \( \|f(\cdot + h) - f(\cdot)\|_1 = O(h) \), and hence we can complete the proof by noting boundedness of \( f \) and \( \|f(\cdot + h) - f(\cdot)\|_2^2 \leq 2\|f\|_\infty \|f(\cdot + h) - f(\cdot)\|_1 \).

\(^8\)After the fact that Raikov [77] proved the law of large numbers \( \sum_{k=1}^{N} f(2^k x)/N \to \int_0^1 f(t) \, dt \) a.e. \( x \) for a locally integrable function \( f \) with period 1, and Riesz [78] pointed out that it is an example of the individual Ergodic theorem for binary transform. For non-integer \( \theta > 1 \) it is very difficult to prove the law of large numbers for \( \sum_{k=1}^{N} f(\theta^k x)/N \). Although Bourgain [15] gave the proof when \( \theta \) is an algebraic number, transcendental case is still open. We can prove the law of large numbers from classical results by assuming some regularity conditions on \( f \), it is difficult to prove for every locally integrable function.
Petit [70] also proved that, for almost every \( \theta > 1 \), the central limit theorem (8) holds with \( \sigma = \sigma(\theta, f) = \|f\|_2 \). Although the case of transcendental \( \theta > 1 \) was open, the central limit theorem (8) with \( \sigma = \sigma(\theta, f) = \|f\|_2 \) was proved in [32] by assuming (10) and (12) when \( \theta \) is not a power root of rational number. By applying the almost sure invariance principles proved in Berkes [3], under (10) and \( L^2 - \beta \)-Lipschitz condition \( (\beta > 0) \), i.e.,

\[
\| f(\cdot + h) - f(\cdot) \|_2 = O(h^\beta),
\]

the law of the iterated logarithm

\[
\limsup_{n \to \infty} \frac{1}{\sqrt{2N \log \log N}} \sum_{k=1}^{N} f(\theta^k x) = \sigma(\theta, f), \quad \text{a.e. } x
\]

was proved in [38]. If we do not use Berkes’ result and prove in a way which is presented in [37], we can prove the law of the iterated logarithm under much weaker regularity condition (13).

As example by Erdős-Fortet explains, only by assuming the Hadamard’s gap condition (1) one cannot expect to have these limit theorems. Even in this case Takahashi [84] proved the upper part of the law of the iterated logarithm

\[
\limsup_{N \to \infty} \frac{1}{\sqrt{N \log \log N}} \sum_{k=1}^{N} f(n_k x) < C_q < \infty, \quad \text{a.e. } x
\]

always holds for \( \beta \)-Lipschitz function \( f (\beta > 0) \) satisfying (10). Philipp [72] proved (17) replacing Lipschitz condition by boundedness of variation. Actually Philipp proved much stronger result, which we will explain in detail in the next section.

Assuming \( \beta \)-Lipschitz continuity \( (\beta > 1/2) \), Dhompongsa [20] relaxed the gap condition to Takahashi’s gap condition. Under the same condition Takahashi [88] proved

\[
\limsup_{N \to \infty} \frac{1}{\sqrt{N \log \log N}} \sum_{k=1}^{N} f(n_k x) \leq \|f\|_A, \quad \text{a.e. } x
\]

and succeeded in giving a concrete upper bound estimate for the first time. Here \( \|f\|_A \) is defined by using the Fourier coefficients in the following way:

\[
\|f\|_A = \sum_{\nu=0}^{\infty} |\hat{f}(\nu)|.
\]

Due to the Bernstein theorem in the theory of Fourier series, if \( f \) is \( \beta \)-Lipschitz continuous \( (\beta > 1/2) \) one has \( \|f\|_A < \infty \), and moreover one has an example of \( f \) such that \( \beta = 1/2 \) and \( \|f\|_A = \infty \).

Péter [69] relaxed the condition to \( \| f(\cdot + h) - f(\cdot) \|_2 = O(h^\beta) (\beta > 0) \) and \( \|f\|_A < \infty \). 9

We have, however, an example for which the upper part of the iterated logarithm fails.

9In the assertion of Péter (18), the left hand side is not \( \|f\|_A \) but the quantity which can be \( \sqrt{2} \) times greater than this value. If we apply trivial modification to the proof, we can have (18).
Theorem 7 (Berkes–Philipp [9]) Let \( f(x) = x - \lfloor x \rfloor - 1/2 \) and \( \{\rho_k\} \) be a sequence monotonously decreasing to 0 satisfying the condition \( \sup \rho_k / \rho_{k+2} < \infty \). Then there exists a sequence \( \{n_k\} \) with
\[
n_{k+1}/n_k \geq 1 + \rho_k
\]
such that
\[
\limsup_{N \to \infty} \frac{1}{\log(1/\rho_N) \sqrt{N \log \log N}} \sum_{k=1}^N f(n_kx) \geq D > 0, \quad \text{a.e. } x
\]
Here \( D \) is an absolute constant.

The function \( f \) appearing in the above theorem is discontinuous, and hence we see \( \|f\|_A = \infty \). Having these results, we expect that \( \|f\|_A \) dominates the upper bound of the law of the iterated logarithm, or more precisely, that (17) can be proved only assuming \( \|f\|_A < \infty \). Actually Berkes [12] derived finiteness of the left hand side of (18) from \( \|f\|_A < \infty \). By modifying its proof, we can prove the universality and best possibility of (18).

Theorem 8 ([35])

(i) If (10) and \( \|f\|_A < \infty \) are satisfied, then we have (18).

(ii) Moreover, if \( \arg \hat{f}(\nu) \) does not depend on \( \nu > 0 \), then (18) is the best possible estimate under the Hadamard’s gap condition, i.e., for any \( \varepsilon > 0 \) there exists a sequence \( \{n_k\} \) with Hadamard’s gap (1) which satisfies
\[
\limsup_{N \to \infty} \frac{1}{\sqrt{N \log \log N}} \sum_{k=1}^N f(n_kx) \geq \|f\|_A - \varepsilon, \quad \text{a.e. } x.
\]

If \( \|f\|_A = \infty \), we clearly have (18), i.e., (18) is valid if \( \|f\|_A \) is well defined, and see that it is the universal result under Takahashi’s gap condition. By (ii), it is the best possible estimate for \( f \) with Fourier cosine series or sine series having non-negative coefficients.

We can also see that, if we assume any weaker gap condition, we can construct a sequence which attains the best possible constant.

Theorem 9 ([36]) For any positive sequence \( \{\rho_k\} \) converging to 0, there exists a sequence \( \{n_k\} \) with (19) such that, for any function \( f \) satisfying conditions in (ii) of Theorem 8, it holds that
\[
\limsup_{N \to \infty} \frac{1}{\sqrt{N \log \log N}} \sum_{k=1}^N f(n_kx) = \|f\|_A, \quad \text{a.e. } x.
\]

The studies in case of \( \|f\|_A = \infty \) is not enough yet, and we only have the result below:

Theorem 10 (Berkes-Philipp [13]) Let \( f \) be a function of bounded variation satisfying (10). Suppose that \( \{\rho_k\} \) is a sequence satisfying \( \rho_k \geq k^{-\alpha} \) (\( 0 < \alpha < 1/2 \)) and a sequence of positive integers \( \{n_k\} \) satisfies (19). Then
\[
\limsup_{N \to \infty} \frac{1}{\log(1/\rho_N^2) \sqrt{N \log \log N}} \sum_{k=1}^N f(n_kx) < \infty, \quad \text{a.e. } x.
\]
4 The uniform distribution theory

We say that a sequence \( \{x_k\} \) of real numbers is uniformly distributed modulo 1 if

\[
\frac{1}{N} \# \{n \leq N : (x_k) \in [a',a) \} \to a - a' \quad (N \to \infty)
\]

holds for every \([a',a) \subset [0,1)\), where \((x_k) = x_k - [x_k]\) is a fractional part of \(x_k\). This convergence appears to be uniform in \(a'\) and \(a\), and it is important to estimate discrepancies for error estimate in numerical integration, and it is important to estimate discrepancies for almost every \(x\).

Theorem 11

If \(\{n_k\}\) is a sequence of strictly increasing sequence of integers, then \(\{n_kx\}\) is uniformly distributed modulo 1 for almost every \(x\). Equivalently \(D_N(\{n_kx\}) \to 0\) and \(D^*_N(\{n_kx\}) \to 0\) a.e. \(x\) as \(N \to \infty\).

In general this result holds only for almost every \(x\) and one cannot expect to hold for every \(x\). For example the sequence \(\{k!e\}\) is not uniformly distributed modulo 1, and when \(\{n_k\}\)
satisfies the Hadamard’s gap condition (1), the set of \( x \) for which \( \{n_k x\} \) is not uniformly distributed modulo 1 has Hausdorff dimension 1 ([26]). In this sense we can say that Weyl’s metric result reveals the property which can be found only when we allow the exceptional set of measure zero.

As to the speed of convergence of discrepancies, Cassels [17] and Erdős-Koksma [23] independently derived \( ND_N(\{n_k x\}) = O(\sqrt{N}(\log N)^{3/2+\epsilon}) \) a.e. Moreover, Baker [1] applied the Carleson-Hunt inequality which was invented to prove the almost everywhere convergence of Fourier series of functions belonging to \( L^p \) (\( p > 1 \)), and improved the result to \( ND_N(\{n_k x\}) = O(\sqrt{N}(\log N)^{3/2+\epsilon}) \) a.e.

Let us state inequalities describing relations among discrepancies and trigonometric sums:

\[
\frac{1}{4K N} \sum_{k=1}^{N} e^{2\pi i K x_k} \leq D_N^*(\{x_k\}) \leq \frac{4}{H} + 4 \sum_{h=1}^{H} \frac{1}{h N} \sum_{k=1}^{N} e^{2\pi i h x_k} \quad (23)
\]

While the left inequality is called Erdős-Turán inequality, the right inequality is a special case of Koksma’s inequality when \( f(x) = e^{2\pi i K x} \). By applying the inequalities to a sequence \( \{n_k\} \) satisfying the Hadamard’s gap condition (1) and by noting the law of the iterated logarithm by Erdős-Gál [25], we have

\[
\limsup_{N \to \infty} \frac{ND_N(\{n_k x\})}{\sqrt{2N \log \log N}} \geq \frac{1}{4\sqrt{2}}, \quad \text{a.e. } x.
\]

In Nijenrode Lecture [28] on 1962, Erdős stated that it is possible to prove

\[
ND_N^*(\{n_k x\}) = O(\sqrt{N}(\log \log N)^c), \quad \text{a.e. } x \quad (c \geq 1/2)
\]

under the same gap condition, and conjectured that it is valid for all strictly increasing sequence \( \{n_k\} \) of integers. Baker [1] independently conjectured

\[
ND_N^*(\{n_k x\}) = O(\sqrt{N}(\log N)^c), \quad \text{a.e. } x
\]

for all \( c > 0 \).

Later Fiedler-Jurkat-Körner [30] proved that for non-negative non-decreasing function \( g \), the convergence \( \sum 1/(kg^4(k)) < \infty \) is equivalent to

\[
\sum_{k=1}^{N} e^{2\pi i k^2 x} = o(\sqrt{N}g(N)), \quad \text{a.e. } x.
\]

By applying the inequality (23), we see that \( ND_N^*(\{k^2 x\}) = o(\sqrt{N}(\log N)^{1/4}) \), a.e. is impossible to hold, and both of conjectures by Erdős and Baker is disproved. Relating this problem, Berkes-Philipp [9] constructed a sequence \( \{n_k\} \) such that

\[
ND_N^*(\{n_k x\}) = O(\sqrt{N}(\log N)^{1/2}), \quad \text{a.e. } x
\]

fails to hold and satisfying the gap condition

\[
n_{k+1}/n_k \geq 1 + 1/\sqrt{N}(\log N)^\alpha \quad (\alpha > 0).
\]

According to Harman [45], it took more than ten years to be noticed that conjectures are disproved by the result of Fiedler-Jurkat-Körner. The first who noticed that fact was Kano [55].
They made a conjecture that the Erdős’ conjecture is valid under Takahashi’s gap condition ([11]).

There are only a few sequences \( \{ n_k x \} \) for which the asymptotic behavior of discrepancies are well studied, i.e., \( n_k = k \), the sequences satisfying the Hadamard’s gap condition (1), \( n_k = 2^k \), very rapidly diverging \( n_k \), and the Hardy-Littlewood-Pólya sequences.

First we explain the results on \( \{ k x \} \). Kesten [58] proved the asymptotic behavior in measure as follows:

\[
\lim_{N \to \infty} \left| \frac{N D_N^* (\{ k x \})}{\log N \log \log N} - \frac{2}{\pi^2} > \varepsilon \right| = 0
\]

for all \( \varepsilon > 0 \). It was proved by Khintchine [59] that for increasing function \( g \), \( \sum 1/g(n) < \infty \) if and only if

\[
N D_N^* (\{ k x \}) = O ((\log N) g(\log \log N)), \quad \text{a.e. } x.
\]

Philipp [72] proved the bounded law of the iterated logarithm

\[
\limsup_{N \to \infty} \sqrt{2 N \log \log N} = C, \quad \text{a.e. } x
\]

(24) for a sequence \( \{ n_k \} \) satisfying the Hadamard’s gap condition (1) and \( \sigma^2(\theta, f_{a', a}) \) is the limiting variance of the central limit theorem of Riesz-Raikov sum \( \sum f_{a', a}(\theta^k x) \) with respect \( f_{a', a}(x) = 1_{[a', a)}((x)) - (a - a') \).

Dhompongса [21], however, assumed a very strong gap condition

\[
\frac{\log(n_{k+1}/n_k)}{\log \log k} \to \infty
\]

(25) to prove the strong approximation of empirical processes with Kiefer process. As a corollary, the law of the iterated logarithm

\[
\limsup_{N \to \infty} \frac{N D_N (\{ k^2 x \})}{\sqrt{2N \log \log N}} = \limsup_{N \to \infty} \frac{N D_N^* (\{ k x \})}{\sqrt{2N \log \log N}} = \frac{1}{2}, \quad \text{a.e. } x
\]

(26) can be derived. That is, when a sequence \( \{ n_k \} \) has very large gap (25), behavior of the sequence \( \{ n_k x \} \) imitates that of sequence of uniform independent random variables, and obeys the Chung-Smirnov type theorem.

Later Philipp [75] proved (24) for Hardy-Littlewood-Pólya sequences. Here \( C \) is a constant depending only on \( \tau \). Moreover Berkes-Philipp-Tichy [14] proved (24) under some mild conditions including Hardy-Littlewood-Pólya sequences.

So far we have explained, the law of the iterated logarithm for discrepancies was completely proved only in the case of very large gap which Dhompongса studied, and the result for power of two given by Philipp did not evaluate a concrete value of limsup constant appearing there.

By recalling the limit theorems for Riesz-Raikov sums, it is expected to have the law of the iterated logarithm for discrepancies of \( \{ \theta^k x \} \) for general \( \theta > 1 \), and actually we have the following result.
Theorem 12 ([38]) For every $\theta > 1$, the law of the iterated logarithm
\[
\limsup_{N \to \infty} \frac{N D_N((\theta^k x))}{\sqrt{2N \log \log N}} = \limsup_{N \to \infty} \frac{N D_N^\ast((\theta^k x))}{\sqrt{2N \log \log N}} = \sup_{0 \leq a < 1} \sigma(\theta, f_{0,a}) := \Sigma_\theta, \text{ a.e. } x,
\]
holds. Here the constant $\Sigma_\theta$ satisfies the properties below:

(i) If $\theta$ is not a power root of any rational number, i.e., if $\theta^r \notin \mathbb{Q}$ ($r \in \mathbb{N}$), then $\Sigma_\theta = 1/2$.

(ii) If $\theta$ is a power root of a rational number, if we determine $p$, $q$ and $r$ by (14), we have $1/2 \leq \Sigma_\theta \leq \sqrt{(pq+1)/(pq-1)/2}$.

(iii) Moreover if $p$ and $q$ are both odd numbers, then $\Sigma_\theta = \sqrt{(pq+1)/(pq-1)/2}$.

(iv) Especially, if $p$ is odd and $q = 1$, then $\Sigma_\theta = \sqrt{(p+1)/(p-1)/2}$.

(v) If $p \geq 4$ is even and $q = 1$, then $\Sigma_\theta = \sqrt{(p+1)p(p-2)/(p-1)^2/2}$.

(vi) If $p = 2$ and $q = 1$, then $\Sigma_\theta = \sqrt{32}/9$.

When $\theta$ is not a power root of any rational numbers, the law of the iterated logarithm for discrepancies of $\{\theta^k x\}$ appears to be completely the same as that of the sequences of uniformly distributed independent random variables. And when $\theta$ is a power root of a positive integer or a rational number which is a ratio of odd integers, concrete values of $\Sigma_\theta$ are calculated. Among other cases, a concrete value of $\Sigma_\theta$ is known only in the case of a power root of $5/2$, and for other $\theta$, is not known.

We can actually have stronger result than (ii), and say that $\Sigma_\theta$ is strictly greater than $1/2$ when $\theta$ is a power root of a rational number. Combining it with (ii), and regarding $\Sigma_\theta$ as a function of $\theta > 1$, it is discontinuous at any point of power of rational number, and is continuous elsewhere.

We can also conclude by the estimate (ii) and the concrete value of $\Sigma_2$ in (vi), that $\Sigma_\theta$ is maximum at $\theta$ of power root of $2$. Hence among all of $\{\theta^k x\}$, the sequences given by $\theta$ of power root of $2$ are furthest form the uniform distribution.

We now explain mechanism why we can prove the law of the iterated logarithm. By using the law of the iterated logarithm for Riesz-Raikov sums (16), heuristic argument gives
\[
\limsup_{N \to \infty} \frac{N D^\ast_N((\theta^k x))}{\sqrt{2N \log \log N}} = \limsup_{N \to \infty} \sup_{0 \leq a < 1} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^{N} f_{0,a}(\langle \theta^k x \rangle) \right|
\]
\[
= \sup_{0 \leq a < 1} \frac{1}{\sqrt{2N \log \log N}} \sum_{k=1}^{N} f_{0,a}(\langle \theta^k x \rangle)
\]
\[
= \sup_{0 \leq a < 1} \sigma(\theta, f_{0,a}).
\]

\[11\] We have $\Sigma^\ast_\theta \geq V(a,a) + 2V(\langle pa \rangle, \langle qa \rangle)/pq$ and we have $\langle pa \rangle \uparrow 1$ and $\langle qa \rangle \uparrow 1/2$ as $a \uparrow 1/2$. The right derivative of the left hand side satisfies $(1-2a) + 2(q(1 - \langle pa \rangle) - \langle qa \rangle)p/q - \rightarrow -1/2q < 0$ as $a \uparrow 1/2$, and hence we see that the left hand side is monotonously decreasing in a small enough left neighborhood of $1/2$. Since it equals to $1/4$ at $a = 1/2$, we can find an $a$ in left neighborhood of $1/2$ such that the value of the left hand side is strictly greater than $1/4$. 

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In the same way, we have
\[
\limsup_{N \to \infty} \frac{ND_N(\{\theta^k x\})}{\sqrt{2N \log \log N}} = \sup_{0 < a' \leq a < 1} \sigma(\theta, f_{a',a}).
\]

To make the above arguments rigorous, the most difficult part is an exchange of two limiting procedures \(\limsup\) and \(\sup\), which is very hard in general. We can justify this part as below.

Take a positive integer \(L\) arbitrarily, and make an approximation in the following way.

\[
\left| \sup_{a < 1} \sum_{k=1}^{N} f_{0,a}(\theta^k x) - \max_{I=1,\ldots,2^{L-1}} \sum_{k=1}^{N} f_{0,2^{-L}I_1}(\theta^k x) \right| \leq \max_{I=1,\ldots,2^{L-1}} \sup_{a < 2^{-L} \cdot 2^{-L+1}} \sum_{k=1}^{N} f_{2^{-L}I_1,2^{-L}I_1+a}(\theta^k x) \right|
\]

By following the proof given by Philipp [72] to have (24), we can prove
\[
\limsup_{N \to \infty} \sup_{a < 2^{-L}} \frac{1}{\sqrt{2N \log \log N}} \sum_{k=1}^{N} f_{2^{-L}I_1,2^{-L}I_1+a}(\theta^k x) \leq C2^{-L/8}.
\]

Here upper bound \(C2^{-L/8}\) is small because the \(L^2\)-norm of the function \(f_{2^{-L}I_1,2^{-L}I_1+a}\) appearing here equals to \(2^{-L/2}\) and is small. It gives an effective approximation, and on the other hand for individual \(I\), by applying the law of the iterated logarithm (16) for Riesz-Raikov sums, we have
\[
\limsup_{N \to \infty} \left| \frac{1}{\sqrt{2N \log \log N}} \sum_{k=1}^{N} f_{0,2^{-L}I_1}(\theta^k x) \right| = \sigma(\theta, f_{0,2^{-L}I_1}).
\]

By taking the maximum for \(I < 2^L\), by applying \(\limsup \max = \max \limsup\), we have
\[
\limsup_{N \to \infty} \max_{I=1,\ldots,2^L-1} \left| \frac{1}{\sqrt{2N \log \log N}} \sum_{k=1}^{N} f_{0,2^{-L}I_1}(\theta^k x) \right| = \max_{I=1,\ldots,2^L-1} \sigma(\theta, f_{0,2^{-L}I_1}).
\]

By letting \(L \to \infty\), we have a rigorous proof of the heuristic argument above.

When \(\theta\) is not a power root of any rational number, we have \(\sigma^2(\theta, f_{a',a}) = \|f_{a',a}\|_2^2 = (a - a') - (a - a')^2\), and hence
\[
\sup_{0 \leq a' < a < 1} \sigma^2(\theta, f_{a',a}) = \sup_{0 \leq a < 1} \sigma^2(\theta, f_{0,a}) = 1/4
\]
is derived, and \(\Sigma_\theta = 1/2\) is verified.

In the above approximation argument, we did not use the concrete structure of the sequence \(\theta^k\), and can be applied to any sequence \(\{n_k\}\) satisfying the Hadamard’s gap condition (1). By applying the law of the iterated logarithm (9) we can prove the following result. It weaken the gap condition assumed by Dhompongsa to the large gap condition, which is optimal gap condition to have the same result as that of sequence of uniformly distributed independent random variables.

**Theorem 13 ([40])** We have the law of the iterated logarithm for discrepancies (26) by assuming the large gap condition (2).
We assume that \( \theta \) is a power root of a rational number and prove the following series expression formula of \( \sigma(\theta, f_{a',a}) \) to prove the assertion of Theorem 12 in this case.

\[
\sigma^2(\theta, f_{a',a}) = \tilde{V}(a', a, a', a) + 2 \sum_{k=1}^{\infty} \frac{\tilde{V}(\langle p^k a', \langle p^k a, \langle q^k a \rangle \rangle)}{p^k q^k}
\]

where

\[
V(x, \xi) = x \wedge \xi - x \xi \quad \tilde{V}(x, y, \xi, \eta) = V(x, \xi) + V(y, \eta) - V(x, \eta) - V(y, \xi).
\]

It is given by calculating the limiting variance (15) of Riesz-Raikov sums. Although it was originally proved by complicated calculation, we here give an elegant proof given by Shiga [81]. Let \( \{ B_t \} \) be a brownian motion. For \( 0 \leq x, y, \xi, \eta \leq 1 \)

\[
\tilde{V}(x, y, \xi, \eta) = E((B_x - B_y)(B_\xi - B_\eta) - (x - y)(\xi - \eta)).
\]  

(28)

We assume \( B_0 = 0 \) and denote \( \int_0^1 f(t) dB_t \) symbolically by \( (f, B) \).

**Lemma 1** If \( 0 \leq a' \leq a < 1 \), and if \( \mu \) and \( \nu \) are relatively prime each other,

\[
\int_0^1 f_{a',a}(\mu x)f_{a',a}(\nu x) = \frac{1}{\mu \nu} \tilde{V}(\langle \mu a' \rangle, \langle \nu a \rangle, \langle \mu a' \rangle, \langle \nu a \rangle).
\]

For \( 0 \leq a', a < 1 \), put \( I_{a',a}(x) = 1_{[0,a]}(\langle x \rangle) - 1_{[0,a']}(\langle x \rangle) \). By denoting \( e_n(x) = e^{2\pi i n x} \), since we have \( e_n(\langle \mu x \rangle) = e_{\mu n}(x) \), the Fourier series expansion \( I_{a',a} = a - a' + \sum_{n \neq 0}(e_n, I_{a',a})e_n \) and the Parseval equality yield

\[
\int_0^1 f_{a',a}(\mu x)f_{a',a}(\nu x) dx = \int_0^1 I_{a',a}(\mu x)I_{a',a}(\nu x) dx - (a - a')^2
\]

\[
= \sum_{m,n \neq 0, \mu = \mu, \nu = \nu}(e_n, I_{a',a})(\overline{e_m}, I_{a',a}) = \sum_{k \neq 0}(e_{\mu k}, I_{a',a})(\overline{e_{\nu k}}, I_{a',a})
\]

\[
= \frac{1}{\mu \nu} \sum_{k \neq 0}(e_{\mu k}, I_{\langle \mu a' \rangle})(\overline{e_{\nu k}}, I_{\langle \nu a \rangle}).
\]

Here, in the last equality we used the relation \((e_{\mu k}, I_{a',a}) = (e_k, I_{\langle \nu a' \rangle, \langle \nu a \rangle})/\nu \). On the other hand, we have

\[
B_a - B_{a'} = (I_{a',a}, B) = (a - a')B_1 + \sum_{n \neq 0}(e_n, I_{a',a})(e_n, B),
\]

and hence by noting (28) and \( E((e_n, B)(\overline{e_m, B})) = \delta_{n,m} \) we have

\[
\tilde{V}(x, y, \xi, \eta) = \sum_{n \neq 0}(e_n, I_{x,y})(e_n, I_{\xi,\eta}).
\]

By combining these we have the conclusion.

To prove the theorem, we have to verify

\[
\sup_{0 < a' \leq a < 1} \sigma(\theta, f_{a',a}) = \sup_{0 < a < 1} \sigma(\theta, f_{0,a}).
\]

Since clearly the right hand side is bounded by the left hand side, we have to prove the converse inequality. By noting the series expression (27), it is sufficient to prove the next lemma.
Lemma 2 We have \( \tilde{V}(\langle x \rangle, \langle y \rangle, \langle \xi \rangle, \langle \eta \rangle) \leq \tilde{V}(0, \langle y - x \rangle, 0, \langle \eta - \xi \rangle) = V(\langle x - y \rangle, \langle \xi - \eta \rangle) \).

As to this lemma also, we here state a simple proof by Shiga [81].

We note that \( \langle -x \rangle = 1 - \langle x \rangle \) and that \( \langle x \rangle \geq \langle y \rangle, \langle x \rangle - \langle y \rangle = \langle x - y \rangle \). We denote the length \( a - a' \) of interval \([a', a]\) by \([a', a]\), and we use the notation \([a', a]^c\) to denote the complement with respect to \([0, 1]\).

By using, \(28\), when \( \langle x \rangle \geq \langle y \rangle \) and \( \langle \xi \rangle \geq \langle \eta \rangle \), we have

\[
\tilde{V}(\langle x \rangle, \langle y \rangle, \langle \xi \rangle, \langle \eta \rangle) = \left[ \langle y \rangle, \langle x \rangle \right] \cap \left[ \langle \eta \rangle, \langle \xi \rangle \right] - \langle x - y \rangle \langle \xi - \eta \rangle
\leq \left[ \langle y \rangle, \langle x \rangle \right] \cap \left[ \langle \eta \rangle, \langle \xi \rangle \right] - \langle x - y \rangle \langle \xi - \eta \rangle = V(\langle x - y \rangle, \langle \xi - \eta \rangle).
\]

We can also prove

\[
\tilde{V}(\langle x \rangle, \langle y \rangle, \langle \xi \rangle, \langle \eta \rangle) = \left[ \langle y \rangle, \langle x \rangle \right] - \left[ \langle y \rangle, \langle x \rangle \right] \cap \left[ \langle \eta \rangle, \langle \xi \rangle \right] - \langle x - y \rangle \langle \xi - \eta \rangle \geq \left[ \langle y \rangle, \langle x \rangle \right] - \left[ \langle y \rangle, \langle x \rangle \right] \cap \left[ \langle \eta \rangle, \langle \xi \rangle \right] - \langle x - y \rangle \langle \xi - \eta \rangle = -V(\langle x - y \rangle, \langle \eta - \xi \rangle),
\]

and hence when \( \langle x \rangle \geq \langle y \rangle \) and \( \langle \xi \rangle < \langle \eta \rangle \), we also have

\[
\tilde{V}(\langle x \rangle, \langle y \rangle, \langle \xi \rangle, \langle \eta \rangle) = -\tilde{V}(\langle x \rangle, \langle y \rangle, \langle \eta \rangle, \langle \xi \rangle) \leq V(\langle x - y \rangle, \langle \xi - \eta \rangle).
\]

The other cases can be reduced to the above cases by noting

\[
\tilde{V}(\langle x \rangle, \langle y \rangle, \langle \xi \rangle, \langle \eta \rangle) = \tilde{V}(\langle \xi \rangle, \langle \eta \rangle, \langle x \rangle, \langle y \rangle).
\]

By these considerations, we have

\[
\Sigma_2^2 = \sup_{0 \leq a < 1} \sigma^2(\theta, f_{0,a}) = \sup_{0 \leq a < 1} \left( V(a, a) + 2 \sum_{k=1}^{\infty} \frac{V((p^k a), (q^k a))}{p^k q^k} \right)
\]

by using a special case of the formula \((27)\) when \( a' = 0 \). Since \( \sigma^2(\theta, f_{0,a}) \) is a continuous function which is symmetric with respect the point \( a = 1/2 \) that is \( \sigma^2(\theta, f_{0,a}) = \sigma^2(\theta, f_{0,1/2-a}) \), we have to calculate the maximum of \( \sigma^2(\theta, f_{0,a}) \) for \( 0 \leq a \leq 1/2 \). By regarding

\[
V(x, \xi) = \begin{cases} 
  x(1 - \xi) & (0 \leq x \leq \xi \leq 1), \\
  (1 - x)\xi & (0 \leq \xi \leq x \leq 1), 
\end{cases}
\]

as a function of \( x \), \( V(x, \xi) \) is increasing for \( x < \xi \), is decreasing for \( x > \xi \), and takes its maximum at \( x = \xi \). Hence we have

\[
0 \leq V(x, \xi) = x \wedge \xi - x \xi \leq V(\xi, \xi) = \xi - \xi^2 \leq \frac{1}{4} = V(1/2, 1/2).
\]

By applying this estimate, we have the upper estimate of \( \Sigma_\theta \) as below:

\[
\sigma^2(\theta, f_{0,a}) \leq 1 + 2 \sum_{n=1}^{\infty} \frac{1}{4^p q^q} = \frac{pq + 1}{4(pq - 1)}
\]

Since each summand in \((29)\) is non-negative, we have the lower bound of \( \Sigma_\theta \) as below:

\[
\sigma^2(\theta, f_{0,1/2}) \geq V(1/2, 1/2) = \frac{1}{4}.
\]
When both of \( p \) and \( q \) are odd, since \( 1/2 \) is a fixed point of both of mappings \( a \mapsto pa \mod 1 \) and \( a \mapsto qa \mod 1 \), we have \( \langle p^{n/2} \rangle = \langle q^{n/2} \rangle = 1/2 \) and

\[
\sigma^2(\theta, f_{0,1/2}) = V(1/2, 1/2) + 2 \sum_{n=1}^{\infty} \frac{V(1/2, 1/2)}{p^n q^n} = \frac{pq + 1}{4(pq - 1)},
\]

which concludes that the value of \( \Sigma^2_\theta \) equals to this value.

When one of \( p \) or \( q \) are even, \( 1/2 \) is not a fixed point and this calculation fails to hold. The maximum is taken at the point other than \( 1/2 \). When \( q = 1 \) and \( p \geq 4 \) is even, the maximum is taken at \( a = a_p = (p/2 - 1)/(p - 1) \) which is the largest fixed point less than \( 1/2 \). When \( q = 1 \) and \( p = 2 \), there is no fixed point other than \( 0 \), and the maximum is taken at \( 1/3 \) which is not a fixed point but a point with period 2. And when \( q > 1 \), analysis becomes much harder.

We can find the full proof in [38] and a graphical explanation in [39].

**Acknowledgement** The author thank the referee for valuable comments.

**References**


[23] P. Erdős and J. Koksma, On the uniform distribution modulo 1 of sequence $(f(n, \theta))$, Indagation Math., 11 (1948) 299-302


[28] P. Erdős, Problems and results on diophantine approximations, Compos. Math., 16 (1964) 52-65


[57] T. Kawata, Gap theorems in Fourier analysis (In Japanese), Seminar on Mathematical Sciences, Keio University, No. 1 (1980)


[72] W. Philipp, Limit theorems for lacunary series and uniform distribution mod 1, Acta Arith., \textbf{26} (1975) 241-251


(Received 2008 Feb. 3)
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