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Feynman’s proof and non-elastic displacement fields: Relationship between magnetic field and defects field

Nozomu Nakamura · Kazuhito Yamasaki

Abstract We consider the relationship between the magnetic field and the non-elastic displacement field including defects, from the viewpoints of non-commutativity of the positions and non-commutativity of the derivatives. The former non-commutativity is related to the magnetic field by Feynman’s proof (1948), and the latter is related to the defect fields by the continuum theory of defects. We introduce the concept of differential geometry to the non-elastic displacement field and derive an extended relation that includes basic equations, such as Gauss’s law for magnetism and the conservation law for dislocation density. The relation derived in this paper also extends the first Bianchi identity in linear approximation to include the effect of magnetism. These findings suggest that Feynman’s approach with a non-elastic displacement field is useful for understanding the relationship between magnetism and non-elastic mechanics.

Keywords Feynman’s proof · magnetism · defects field · non-elastic displacement · differential geometry

1 Introduction

In 1948, Feynman proved Maxwell’s equations of electromagnetism, assuming commutation relations between position and velocity of a single non-relativistic particle obeying Newton’s law of motion (e.g., [7,28]). This proof,
although based on simple mathematical assumptions, gives rise to nontrivial generalizations [4]. Since Dyson’s paper (1990) [7], Feynman’s scheme has been extended to the physical and mathematical fields such as special and general relativity, non-Abelian gauge theory, and non-commutative geometry (e.g., [4, 6, 20, 21, 26, 27]).

As mentioned above, the concept of the commutation relation is central to deriving Maxwell’s equations in Feynman’s proof. On the other hand, as discussed in Sect. 3, introducing the concept of the commutation relation to the non-elastic displacement field simplifies the expression of the basic equations for defect fields. Magnetic-deformation interactions, such as magnetostrictive and Villari effects (e.g., [5]), have been applied in not only engineering fields (e.g., [10]) but also earth and planetary sciences [3, 23–25]. In this case, non-elastic deformation, such as the displacement field with defects, plays a critical role (e.g., [3, 23]). However, in contrast to the gauge field (e.g., electromagnetic and gravity fields), there has been no detailed investigation of applying Feynman’s approach to non-elastic displacement fields that include defects. Therefore, a systematic understanding of how such non-elastic mechanics affects the magnetic phenomena through the commutation relation is still lacking. The purpose of this paper is to unify the magnetic and non-elastic (defects) fields in a systematic way using commutation relations.

For this analysis, we introduce the concept of differential geometry to Feynman’s approach, because it is well known that the defect field resulting from dislocations and disclinations can be expressed by differential geometrical objects, such as the torsion tensor and curvature tensor caused by the non-elastic displacement field (e.g., [9, 17, 31–33]). It should be noted that this differential geometrical approach is not physically related to the previous one, in terms of extending Feynman’s proof from the viewpoints of the general theory of relativity or the non-Abelian gauge theory. The differential geometry used in this paper is related to modern continuum mechanics for defect fields, referred to as the continuum theory of defects [17, 30].

The structure of this paper is as follows. In Sect. 2, we review Feynman’s proof, particularly noting Gauss’s law for magnetism. In Sect. 3, the basic equations for defect fields are rewritten, based on the commutation relations. In Sect. 4, the results of Sect. 2 and Sect. 3 are unified to derive the relationship between the magnetic field and the defects field. Although not considered in the present work, in Sect. 5, we comment on the electric field and dislocation flow.

2 Brief Review of Feynman’s proof: the Gauss’s law for magnetism

In this section, we review Feynman’s proof of Maxwell’s equations in the classical form. The original version of Feynman’s proof, of course, derives not only Gauss’s law but also the Maxwell-Faraday equation (the generalized form of Faraday’s law of induction) [7]. However, because this paper considers the
relations between the magnetic field and the defects field, we focus on the derivation of Gauss's law for magnetism in Feynman’s proof.

Let us consider a particle with position \( x_i \) and velocity \( \dot{x}_i = dx_i/dt \). It obeys the equation of motion:

\[
\ddot{x} = f_i,
\]

and its position and velocity satisfy the commutation relations:

\[
[x_j, x_k] = 0,
\]

\[
[x_j, \dot{x}_k] = i\delta_{jk},
\]  

where the coefficient is ignored for simplicity. The bracket \([\cdots, \cdots]\) possesses the following properties: bilinearity, antisymmetry, the Jacobi identity, and Leibniz rules. From (2) and (3), we obtain a useful formula for function \( f(x, t) \):

\[
[\dot{x}_j, f(x, t)] = -i\partial_j f(x, t),
\]

where \( \partial_i = \partial/\partial x_i \).

Under the above assumptions, Gauss’s law for magnetism can be derived as follows. Differentiating the bracket (3) with respect to time and using (1), we obtain \( [x_j, f_k] + [\dot{x}_j, \dot{x}_k] = 0 \). From the antisymmetry for the index, we may write:

\[
[\dot{x}_j, \dot{x}_k] = -[x_j, f_k] = i\epsilon_{ijk} B_i.
\]

This is recognized as the definition of magnetic field \( B_i \), rewritten as:

\[
B_l = -(i/2)\epsilon_{lmn}[\dot{x}_m, \dot{x}_n].
\]

From the Jacobi identity and the relation (3), Eq. (5) leads to:

\[
[x_l, B_l] = 0.
\]

This means that \( B_i \) is a function of \( x \) and \( t \) only. Therefore, we can use the formula (4): \( [\dot{x}_l, B_l] = -i\partial_l B_l \). On the other hand, the Jacobi identity and (5) give \( [\dot{x}_l, B_l] = 0 \). Therefore, we obtain Gauss’s law for magnetism:

\[
\partial_l B_l = 0.
\]
3 Continuum theory of defects in terms of the commutation relation

A differential geometrical description of a deformed medium including a defect field was formulated by Kondo in 1952 [15]. Since then, this mathematical approach has been developed on the basis of Riemann-Cartan geometry (e.g., [9]) and applied in several fields, such as earth and planetary sciences (e.g., [22, 29, 34]). Following Kröner [17], we refer to this as the continuum theory of defects. It should be noted that our differential geometrical approach is not physically related to the previous one as an extension of Feynman’s proof from the viewpoint of the general theory of relativity (e.g., [28]). Our geometrical approach is simply related to modern continuum mechanics for the defects field.

In the continuum theory of defects, a deformed medium is described by two kinds of space, each with different choices of metrics [1, 2, 13, 18, 31]: strain space, for which distortion (gradient of displacement) is chosen as the metric; and stress space, for which the stress function is chosen as the metric. The Hodge star operator, in differential form, links the strain space to the stress space throughout the Hodge dual relations that correspond to the constitutive equation [32, 33]. Here, the basic equation in strain space is rewritten, using the commutation relations to relate the defects field to the magnetic field already formalized by the commutation relation.

Let us consider the displacement field $u_i$ including defects. The shifted position $\tilde{x}_i$ is given by:

$$x_i \rightarrow \tilde{x}_i = x_i + u_i,$$

where $u_i = u_i(x_i, t)$. In this case, the basic tetrads related to the physical coordinates of material points $x^a$ are [14]:

$$e_i^a = \delta^i_a - \partial_a u^i,$$

$$e^a_i = \delta^a_i + \partial_i u^a.$$

Based on the basic tetrads, we can obtain the other important quantities in the differential geometry: metric tensor $g_{ij}$, connection $\Gamma_{ijk}$, torsion tensor $S_{ijk}$, and curvature tensor $R_{ijkl}$, as follows:

$$g_{ij} = e_a^i e^a_j,$$

$$\Gamma_{ijk} = e_a^i \partial_j e^a_k,$$

$$S_{ijk} = \frac{1}{2} (e_a^i \partial_j e^a_k - e_a^j \partial_i e^a_k),$$

$$R_{ijkl} = \frac{1}{2} e_a^i (\partial_k \partial_j e^a_l - \partial_l \partial_k e^a_j).$$
From (10) and (11) in linear approximation, these geometric quantities can be expressed in terms of the displacement field [14]:

\[ g_{ij} = \delta_{ij} + \partial_i u_j + \partial_j u_i, \quad (16) \]

\[ \Gamma_{ijk} = \partial_j \partial_k u_i, \quad (17) \]

\[ S_{ijk} = \frac{1}{2} (\partial_j \partial_k - \partial_k \partial_j) u_i, \quad (18) \]

\[ R_{ijkl} = \frac{1}{2} (\partial_k \partial_l - \partial_l \partial_k) \partial_i u_j. \quad (19) \]

Equations (18) and (19) show that the torsion and curvature are given by the non-commutativity of derivatives against the non-elastic displacement field [14]. In the previous derivation of the relation, corresponding to (18) and (19), we sometimes used the homotopy operator in its differential form [31–33]. In contrast to this formulation, here we use the non-commutative expressions (18) and (19) following Kleinert [14], due to the following considerations.

The torsion and curvature caused by the non-elastic displacement field are closely related to the defects. This correspondence has been studied from various viewpoints of gauge theory in modern continuum mechanics. For instance, in mathematical fields, Edelen and Golebiewska-Lasota demonstrated that the differential geometrical theory of defects admits a 45-fold Abelian gauge condition, called the Golebiewska gauge [8, 9, 11, 12]. In physical field theory, the Kondo-Minagawa gauge condition in the defects field [16] can be physically interpreted as the criterion for yielding related to defects [32]. In these gauge approaches, the dislocation density \( \alpha_{ij} \) and \( \theta \) disclination \( \theta \) (rotational dislocation) density \( \theta_{imj} \) have been recognized as typical examples of defects. The concrete expression of the correspondence between them and geometrical objects are as follows (e.g., [15, 17, 31]):

\[ \alpha_{ij} = \epsilon_{ikl} S_{jkl}, \quad (20) \]

\[ \theta_{imj} = \epsilon_{mkl} R_{ijkl}. \quad (21) \]

Therefore, from (18) and (19), we can express the defects field in terms of the non-elastic displacement field using the commutation relations:

\[ \alpha_{lj} = \frac{1}{2} \epsilon_{lmn} [\partial_m, \partial_n] u_j, \quad (22) \]

\[ \theta_{kjl} = \frac{1}{2} \epsilon_{lmn} [\partial_m, \partial_n] \partial_k u_j. \quad (23) \]
4 Relationship between magnetic fields and the defects field

In this section, we consider the relationship between magnetic fields and the field resulting from defects based on the results in Sect. 2 and Sect. 3. First, the magnetic field obeys Eq. (7) in Sect. 2: \([x_j, B_l] = 0\). Next, we introduce the displacement field obeying (9): \(\tilde{x}_i = x_i + u_i\). In this case, \([\tilde{x}_j, B_l]\) gives the interaction term between the displacement field and the magnetic field, such as \([u_j, B_l]\). We will expand such interaction terms. From (6): \(B_l = - \frac{i}{2} \epsilon_{lmn} [\dot{x}_m, \dot{x}_n]\), the term \([u_j, B_l]\) becomes:

\[
[u_j, B_l] = - \frac{i}{2} \epsilon_{lmn} [u_j, [\dot{x}_m, \dot{x}_n]].
\] (24)

From the Jacobi identity, the right side can be rewritten as:

\[
[u_j, B_l] = \frac{i}{2} \epsilon_{lmn} ([\dot{x}_m, [u_j, \dot{x}_n]] + [\dot{x}_n, [u_j, \dot{x}_m]]) = \frac{i}{2} \epsilon_{lmn} [\partial_m, \partial_n] u_j,
\] (25)

where we use the formula (4) in the last step. In a similar fashion, the interaction term between the distortion (gradient of the displacement) and magnetic field is given by:

\[
[\partial_k u_j, B_l] = \frac{i}{2} \epsilon_{lmn} [\partial_m, \partial_n] \partial_k u_j.
\] (26)

Therefore, from (22) and (23), the terms \([u_j, B_l]\) and \([\partial_k u_j, B_l]\) are related to the dislocation field and the disinclination field, respectively:

\[
[u_j, B_l] = i \alpha_{lj},
\] (27)

\[
[\partial_k u_j, B_l] = i \theta_{kjl}.
\] (28)

Moreover, by partial derivation of (27) with respect to \(x_k\) and using (28), we can obtain the relationship between the defects field and the magnetic field:

\[
\partial_k \alpha_{lj} = \theta_{kjl} - i[u_j, \partial_k B_l].
\] (29)

Equation (29) unifies the results in Sections 2 and 3 through the commutation relations. Therefore, it is expected that Eq. (29) includes the previous basic equations for the defects field and the magnetic field. We consider the case \(k = l\) in (29): \(\partial_l \alpha_{lj} = \theta_{lj} - i[u_j, \partial_l B_l]\). If \(\partial_l B_l = 0\), we obtain \([u_j, \partial_l B_l] = 0\), i.e., \(\partial_l \alpha_{lj} = \theta_{lj}\). Conversely, if \(\partial_l \alpha_{lj} = \theta_{lj}\), we obtain \([u_j, \partial_l B_l] = 0\). This is satisfied for any \(u\) with \(\partial_l B_l = 0\). Therefore, when \(k = l\) in (29), the equation given by

\[
\partial_l B_l = 0,
\] (30)

is accompanied by

\[
\partial_l \alpha_{lj} = \theta_{lj}.
\] (31)
Equation (30) is one of Maxwell’s equations. Equation (31) is also one of the basic equations of the defects field, specifically, the conservation law for dislocations density in the continuum theory of defects (e.g., [14, 32]).

Finally, we reconsider Eq. (29) from the viewpoint of differential geometry. From the relations (20) and (21), Eq. (29) can be expressed in terms of the geometrical objects:

\[ \epsilon_{lmn}(\partial_k S_{jmn} - R_{kjmnl}) = i[u_j, \partial_k B_l]. \]  

(32)

Interestingly, this extends the first Bianchi identity in linear approximation to include the effect of term \([u_j, \partial_k B_l]\). In other words, Eq. (29) also includes the basic equation for differential geometry. In gauge theory, the magnetic field also corresponds to the geometrical quantities. Therefore, Eq. (32) implies that the interaction between the magnetic field and the defects field can be interpreted as the geometric relation among the geometrical quantities.

5 Relationships between electric field and defects field: Future works

This paper focuses on the relationships between the magnetic field and the defects field. The electric field is also known to be related to non-elastic deformation accompanied with the defects field (e.g., [22]). In this section, we comment on the relationships between the electric field and the defects field.

As mentioned in Sect. 2, the original version of Feynman’s proof derived the basic equation for the electric field as follows:

\[ \partial_t B_i + \epsilon_{ijk} \partial_j E_k = 0. \]  

(33)

Needles to say, the time derivative of the magnetic field is related to the electric field. In fact, the derivation of (33) in Feynman’s proof starts at the time derivative of the magnetic field (6).

Now, it is known that the time derivative of the dislocations field yields dislocation flow \(I_{ij}\) (e.g., [32]):

\[ \partial_t \alpha_{il} + \epsilon_{ijk} \partial_j I_{kl} = 0. \]  

(34)

By comparison with (33) and (34), it is expected that the electric field \(E_i\) is related to the defects field through dislocation flow \(I_{ij}\), similar to the relationship between the magnetic field \(B_i\) and the defects via the dislocation density \(\alpha_{ij}\) (27). However, the derivation of dislocation flow, in terms of the non-commutativity of the derivatives, similar to that shown in Sect. 3, has yet to be shown. This algorithm is required in our formulation to connect the electromagnetic field with the defects, so this is an area for our future research.
References